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Z' and the Appelquist–Carrazzone decoupling

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Abstract. We consider the electroweak theory with an additional neutral vector boson Z' at one loop. We propose a renormalization scheme which makes the decoupling of heavy Z' effects manifest. The proposed scheme justifies the usual procedure of performing fits to the electroweak data by combining the full SM loop corrections to observables with the tree-level corrections due to the extended gauge structure. Using this scheme we discuss in the model with extra an $U(1)'$ group factor one-loop results for the ρ parameters defined in several different ways.

1 Introduction

For various reasons new physics is expected to show up at the TeV scale. One of the possibilities, not the least likely one, is that extra gauge boson with masses ~ 1 TeV should be discovered. They are predicted by various string inspired models as well as by some models aiming at solving the hierarchy problem of the SM. Here belong for example Little Higgs models [1] or models combining supersymmetry with the idea of the Higgs doublet as a pseudo-Goldstone boson [2, 3]. Before the advent of the LHC, the electroweak data are used to constrain parameter spaces of such models.

The standard methodology used in testing models of new physics against the electroweak data is that one combines the full one-loop (and also dominant two-loop) corrections to the relevant observables calculated within the SM with modifications stemming from new physics (new gauge bosons, new fermions, etc.) accounted for at the tree level only. Given that the top quark mass is known fairly well, this allows one to constrain other parameters of these models [4].

However, some doubts have been expressed in the literature [5–7] about the validity of this standard approach in models with extended gauge sector. In particular, it has been argued that this approach is not valid in theories in which at the tree level $\rho \neq 1$ since then the entire structure of loop correction is altered and the Appelquist–Carrazzone decoupling does not hold.

To investigate the problem in more detail we consider in this paper the simplest extension of the SM with an additional $U(1)_E$ gauge group and study the one-loop renormalization of the model.[1] We propose a renormalization scheme in which the Appelquist–Carrazzone decoupling is manifest. It combines the on-shell renormalization for the three input observables for which we conveniently choose $\alpha_{\rm EM}$, $G_{\rm F}$ and M_W with the $\overline{\rm MS}$ scheme for the additional parameters introduced by the extended gauge sector. The final expressions for measurable quantities are such that

- they coincide with the SM expression for $M_{Z'} \to \infty$;
- explicit renormalization scale dependence is only in the $M_{Z'}$ suppressed terms;
- they are scale independent when the RG running of the parameters is taken into account. Tadpoles play the crucial role here.

Our scheme can be contrasted with other renormalization schemes used in the literature in which the explicit decoupling of heavy particles (Z') is lost because also the couplings related to the extended gauge sector (couplings of the $U(1)_E$ gauge boson) are expressed in terms of the low energy observables additional to $\alpha_{\rm EM}$, $G_{\rm F}$ and M_Z (or M_W), like $\sin^2\theta_\ell^{\rm eff}$ or ρ. Our scheme can universally be used for $M_{Z'} \sim M_{Z^0}$ or $M_{Z'} \gg M_{Z^0}$ whereas the other ones are practical only for $M_{Z'} \sim M_{Z^0}$. Indeed, for $M_{Z'} \gg M_{Z^0}$, using e.g. $\sin^2\theta_\ell^{\rm eff}$ as an additional input parameter for fixing the coupling of Z' leads, because of the lack in such a scheme of explicit Appelquist–Carrazzone decoupling, to uncertainties which become larger the larger is the Z' mass. The scheme proposed in this paper allows us to directly constrain by the electroweak data the $\overline{\rm MS}$ running parameters of the extended model at a conveniently chosen renormalization scale μ, with $\alpha_{\rm EM}$, $G_{\rm F}$ and M_W chosen as input observables. Furthermore, for $M_{Z'} \gg M_{Z^0}$ it lends justification to the standard approach to testing such a model against electroweak data and makes it rigorous by specifying what parameters are being constrained.

As an illustration of the use of our renormalization scheme and in order to demonstrate that it leads to explicit Appelquist–Carrazzone decoupling we clarify vari-

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[1] For earlier discussions of the renormalization of the $SU(2) \times U(1)_1 \times U(1)_2$ models see [8, 9].

ous aspects of the ρ parameter(s) in the $SU(2)\times U(1)_1\times U(1)_2$ model. First of all, we discuss in detail various definitions of ρ and the corresponding tree-level results. Interestingly enough, there exists a definition of ρ in terms of the low energy neutral to charged current ratio for neutrino processes which leads to $\rho_{\rm low}=1$ as in the SM. Next, we calculate loop corrections to these different ρ parameters and show that in the renormalization scheme with explicit Appelquist–Carrazzone decoupling the celebrated m_t^2/m_W^2 contribution is always present. The milder, logarithmic dependence on m_t claimed in [5, 6] is an artifact of the renormalization scheme in which there is no explicit Appelquist–Carrazzone decoupling.

We also elucidate some specific technical aspects of a theory with $U(1)_1\times U(1)_2$ group factor related to the mixing of the two corresponding gauge bosons resulting in some peculiarities of the RG running of the $U(1)$ gauge couplings.

The plan of the paper is as follows. In Sect. 2 we recall the general structure of the $U(1)_1\times U(1)_2$ gauge theory and introduce effective charges which allow one to cast the Lagrangian in a simple form. We express the renormalization group equations for the $U(1)$ couplings in terms of these effective couplings. We also introduce the simplest extension of the SM by an extra $U(1)$ group factor (with an $SU(2)$ singlet scalar vacuum expectation value (VEV) breaking the extra $U(1)$) which will serve us as a laboratory to illustrate our main points concerning the loop corrections to electroweak observables. In Sect. 3 we define different ρ parameters, calculate them at tree level in the model introduced in Sect. 2 and show that the leading order contribution of Z' to these parameters can be also obtained in the approach using the Appelquist–Carrazzone decoupling. In Sect. 4 we define our renormalization scheme, and apply it in Sect. 5 to calculate the corrections to the low energy ρ parameter defined in terms of the neutrino processes. In Sect. 6 we illustrate the interplay of the proposed scheme with the renormalization group equations derived in Sect. 2 on the one-loop calculation of the Z^0 mass. Finally, in Sect. 7 we briefly discuss the calculation of the dominant top–bottom contribution to the parameter ρ defined in terms of the Z^0, $W^\pm$ gauge boson masses and $\sin^2\theta^\ell_{\rm eff}$ parametrizing the coupling of the on-shell Z^0 to leptons. Several appendices contain technical details necessary in the analyses presented in the main text.

2 $U(1)_1\times U(1)_2$ gauge theory: couplings and their RG equations

The most general kinetic term for two $U(1)$ gauge fields has the form

$$\mathcal{L}^{\rm kin}=-\frac{1}{4}f^1_{\mu\nu}f^1_{\mu\nu}-\frac{1}{4}f^2_{\mu\nu}f^2_{\mu\nu}-\frac{1}{2}\kappa f^1_{\mu\nu}f^2_{\mu\nu}\,. \qquad (1)$$

κ is a real constant constrained by the condition $|\kappa|<1$. The most general covariant derivative of a matter field ψ_k is

$$\mathcal{D}_\mu=\partial_\mu+{\rm i}\sum_{a=1}^{2}\sum_{b=1}^{2}Y_k^a g_{ab}A^b_\mu\,, \qquad (2)$$

where the constants Y_k^a play the role of the $U(1)$ charges of ψ_k and g_{ab} are the coupling constants (running couplings in the $\overline{\rm MS}$ renormalization scheme). The gauge transformations then are

$$A^a_\mu\to A^a_\mu+\partial_\mu\theta^a\,,$$
$$\psi_k\to\exp\left(-{\rm i}\sum_{a=1}^{2}\sum_{b=1}^{2}Y_k^a g_{ab}\theta^b\right)\psi_k\,. \qquad (3)$$

The existence of a whole matrix g_{ab} of couplings in place of only one gauge couplings per each $U(1)$ group factor is a peculiarity of the theory with multiple $U(1)$'s [10, 11]. Even if not introduced in the original Lagrangian, the last term in (1) and the matrix g_{ab} of couplings are generated in the effective action by radiative corrections.

To have simple forms for the tree-level propagators, it is convenient to work in the basis in which the tree-level kinetic mixing is removed.[2] By expressing the original $A^{1,2}_\mu$ fields in terms of the new fields denoted by A^Y_μ and A^E_μ (because they will play the roles of the weak hypercharge and extra $U(1)$ gauge bosons, respectively), we have

$$A^1_\mu=\frac{1}{\sqrt{2(1+\kappa)}}A^Y_\mu+\frac{1}{\sqrt{2(1-\kappa)}}A^E_\mu\,,$$
$$A^2_\mu=\frac{1}{\sqrt{2(1+\kappa)}}A^Y_\mu-\frac{1}{\sqrt{2(1-\kappa)}}A^E_\mu\,, \qquad (4)$$

and the kinetic cross term disappears (but there will be a counterterm $-(1/2)\delta Z f^E_{\mu\nu}f^Y_{\mu\nu}$) and the general form (2) of the covariant derivative does not change. Thus, for each matter field k there are charges Y_k^E and Y_k^Y and there are four couplings g_{YY}, g_{YE}, g_{EY}, g_{EE}. Only three of them are independent [10]: the $U(1)$ gauge fields can be rotated: $A^Y=\cos\vartheta\tilde{A}^Y-\sin\vartheta\tilde{A}^E$, $A^E=\sin\vartheta\tilde{A}^Y+\cos\vartheta\tilde{A}^E$, without reintroducing the kinetic cross term, and such a rotation induces the corresponding rotations of couplings

$$\begin{pmatrix}\tilde g_{YY}\\ \tilde g_{YE}\end{pmatrix}=\begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}g_{YY}\\ g_{YE}\end{pmatrix}$$
$$\begin{pmatrix}\tilde g_{EY}\\ \tilde g_{EE}\end{pmatrix}=\begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}g_{EY}\\ g_{EE}\end{pmatrix}. \qquad (5)$$

The angle ϑ can be chosen so that one of the four couplings vanishes. It is also easy to check that the combinations

$$g_{EE}g_{YY}-g_{EY}g_{YE}\,,\quad g^2_{EE}+g^2_{EY}\,,$$
$$g_{YE}g_{EE}+g_{EY}g_{YY}\,,\quad g^2_{YY}+g^2_{YE} \qquad (6)$$

are the invariants of the rotations (5).

[2] It is also possible to work with non-diagonal kinetic terms [11, 12].

The renormalization group equations for the couplings g_{ab} can be computed in the standard way [10, 11] with the result

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_{ba} = \frac{1}{16\pi^2} \sum_{c,d,e} g_{bc} \times \left[\frac{2}{3} \sum_f \left(Y_f^d Y_f^e\right) + \frac{1}{3} \sum_s \left(Y_s^d Y_s^e\right) \right] g_{dc} g_{ea} , \tag{7}$$

where the first sum is over left-chiral fermions and the second one over complex scalars of the theory.

As an realistic extension of the SM we consider a theory with the $SU(2)_\mathrm{L} \times U(1)_Y \times U(1)_E$ electroweak symmetry spontaneously broken down to $U(1)_\mathrm{EM}$. The required symmetry breaking is ensured by vacuum expectation values of the $SU(2)$ doublet H and of the singlet S. We assume that S is charged under only one $U(1)$, that is $Y_S^Y = 0$ (but $Y_H^Y \neq 0$ and $Y_H^E \neq 0$), so that $\langle S \rangle = v_S/\sqrt{2}$ leaves unbroken $SU(2)_\mathrm{L} \times U(1)_Y$. It is then convenient to make the orthogonal field redefinition (which does not reintroduce the kinetic mixing term)

$$E_\mu = \frac{g_{EE} A_\mu^E + g_{EY} A_\mu^Y}{\sqrt{g_{EE}^2 + g_{EY}^2}} , \qquad B_\mu = \frac{-g_{EY} A_\mu^E + g_{EE} A_\mu^Y}{\sqrt{g_{EE}^2 + g_{EY}^2}} , \tag{8}$$

where E_μ is the combination which becomes massive after $U(1)_E$ breaking by $v_S \neq 0$, and B_μ will play the role of the weak hypercharge gauge field. The couplings of the generic matter field ψ_k to E_μ and B_μ are then given by

$$g_y Y_k B_\mu + \left(g_E Y_k^E + g' Y_k^Y\right) E_\mu , \tag{9}$$

where

$$g_y \equiv \frac{g_{EE} g_{YY} - g_{EY} g_{YE}}{\sqrt{g_{EE}^2 + g_{EY}^2}} ,$$
$$[4pt] g_E \equiv \sqrt{g_{EE}^2 + g_{EY}^2} ,$$
$$g' \equiv \frac{g_{YE} g_{EE} + g_{EY} g_{YY}}{\sqrt{g_{EE}^2 + g_{EY}^2}} \tag{10}$$

are invariants of the transformations (5). Because only three couplings are physical the last invariant, $g_{YY}^2 + g_{YE}^2$ in (6), which does not enter the definitions of g_y, g_E and g', can be expressed in terms of these:

$$g_{YY}^2 + g_{YE}^2 = g_y^2 + g'^2 . \tag{11}$$

From (9) it follows that Y_k^Y corresponds to the SM hypercharge. We assume therefore, that the factors Y_k^Y are as in the SM, in particular, $Y_H^Y = \frac{1}{2}$. It will also prove convenient to introduce effective charges e_k and to rewrite the couplings of matter fields to the extra gauge boson E_μ in the form

$$g_E e_k \equiv g_E Y_k^E + g' Y_k^Y . \tag{12}$$

With the factors e_k the matter Lagrangian can be written in the naive form (frequently used in the literature [13, 14]) as if there were no mixing of the two $U(1)$ group factors. It is however important to remember that the e_k are just a means to compactly write the couplings. They are not quantum numbers (charges) – except for e_S which is constant. They do run with the scale: their RG running can be determined from the running of g_{EE}, g_{YY}, g_{EY}, g_{YE} and of g_E.

The closed system of the RG equations for the three couplings (10) can be readily derived from the general formula (7). Note that these couplings are defined at any renormalization scale μ in the (rotating) basis in which the kinetic mixing term is absent. Using (11) one finds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} g_E &= A^{EE} g_E^3 + 2A^{EY} g_E^2 g' + A^{YY} g_E g'^2 , \\ \frac{\mathrm{d}}{\mathrm{d}t} g_y &= A^{YY} g_y^3 , \\ \frac{\mathrm{d}}{\mathrm{d}t} g' &= A^{YY} g' \left(g'^2 + 2g_y^2\right) + 2A^{EY} g_E \left(g'^2 + g_y^2\right) \\ &\quad + A^{EE} g_E^2 g' , \end{aligned} \tag{13}$$

where

$$A^{ab} = \frac{2}{3} \sum_f \left(Y_f^a Y_f^b\right) + \frac{1}{3} \sum_s \left(Y_s^a Y_s^b\right) . \tag{14}$$

With the identification of Y_k^Y as SM hypercharges, the running of g_y is exactly as in the SM. This could be expected because of the $U(1)$ Ward identity, which ensures the absence of threshold corrections to g_y when the heavy massive E_μ field is decoupled.

In the calculations presented in the following sections we will need RG equations for the combinations $e_S^2 g_E^2$ and $e_H^2 g_E^2$ defined by (12). Using (13) and (14) these RG can be also expressed in terms of the effective couplings (12):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e_S^2 g_E^2 &= 2 e_S^2 g_E^2 \left(\frac{2}{3} \sum_f (e_f g_E)^2 + \frac{1}{3} \sum_s (e_s g_E)^2 \right) \\ \frac{\mathrm{d}}{\mathrm{d}t} e_H^2 g_E^2 &= 2 e_H^2 g_E^2 \left(\frac{2}{3} \sum_f (e_f g_E)^2 + \frac{1}{3} \sum_s (e_s g_E)^2 \right) \\ &\quad + 4 e_H g_E \\ &\quad \times \left(\frac{2}{3} \sum_f e_f g_E Y_f^Y Y_H^Y + \frac{1}{3} \sum_s e_s g_E Y_s^Y Y_H^Y \right) g_y^2 , \end{aligned} \tag{15}$$

Finally, we recall the formulae derived in [13] for gauge boson masses appearing as a result of the electroweak breaking by $\langle S \rangle = v_S/\sqrt{2}$ and $\langle H^0 \rangle = v_H/\sqrt{2}$. The $W^\pm$ boson mass is given as in the SM by $M_W^2 = \frac{1}{4} g_2^2 v_H^2$, whereas the mass matrix of the neutral gauge bosons in the basis

(B_μ, W^3_μ, E_μ) reads

$$\mathcal{M}^2_{\rm neut} = \begin{pmatrix} \frac{1}{4}g_y^2 v_H^2 & -\frac{1}{4}g_y g_2 v_H^2 & \frac{1}{2}g_y g_E e_H v_H^2 \\ -\frac{1}{4}g_y g_2 v_H^2 & \frac{1}{4}g_2^2 v_H^2 & -\frac{1}{2}g_2 g_E e_H v_H^2 \\ \frac{1}{2}g_y g_E e_H v_H^2 & -\frac{1}{2}g_2 g_E e_H v_H^2 & g_E^2\left(e_H^2 v_H^2 + e_S^2 v_S^2\right) \end{pmatrix}. \quad (16)$$

It is diagonalized by two successive rotations so that the mass eigenstates are given by

$$\begin{pmatrix} B_\mu \\ W^3_\mu \\ E_\mu \end{pmatrix} = \begin{pmatrix} c & -sc' & ss' \\ s & cc' & -cs' \\ 0 & s' & c' \end{pmatrix} \begin{pmatrix} A_\mu \\ Z^0_\mu \\ Z'_\mu \end{pmatrix}, \quad (17)$$

where $c \equiv \cos\theta_{\rm W}$, $s \equiv \sin\theta_{\rm W}$ are as in the SM: $s/c = g_y/g_2$, and $c' \equiv \cos\theta'$, $s' \equiv \sin\theta'$, where

$$\tan 2\theta' = \frac{2\left(-\frac{1}{2}\sqrt{g_y^2+g_2}g_E e_H v_H^2\right)}{\frac{1}{4}\left(g_y^2+g_2^2\right)v_H^2 - g_E^2\left(e_H^2 v_H^2 + e_S^2 v_S^2\right)}. \quad (18)$$

The masses of the two gauge bosons, Z^0 and Z' are given by

$$M^2_{Z^0} = \frac{1}{2}\left(A+B-\sqrt{(A-B)^2+4D^2}\right),$$
$$M^2_{Z'} = \frac{1}{2}\left(A+B+\sqrt{(A-B)^2+4D^2}\right), \quad (19)$$

where $A = M_W^2/c^2$, $B = e_S^2 g_E^2 v_S^2 + e_H^2 g_E^2 v_H^2$ and $D = -(e/2sc)e_H g_E v_H^2$. The electric charge e is given by the same formula as in the SM: $e = g_y c = g_2 c$. In Appendix A we record some formulae which will prove indispensable in various manipulations.

The interactions of the matter fermions with Z^0 and Z' bosons takes the form

$$\mathcal{L}_{\rm int} = -J^\mu_{Z^0} Z^0_\mu - J^\mu_{Z'} Z'_\mu,$$

where the currents are easily found to be

$$J^\mu_{Z^0} = \sum_{f=\nu,e,u,d}\left[\frac{e}{sc}\left(T^3_f - s^2 Q_f\right)c' + e_f g_E s'\right]\bar\psi_f\gamma^\mu \mathbf{P}_{\rm L}\psi_f + \sum_{f=e,u,d}\left[\frac{e}{sc}\left(-s^2 Q_f\right)c' - e_{f^c} g_E s'\right]\bar\psi_f\gamma^\mu \mathbf{P}_{\rm R}\psi_f, \quad (20)$$

$$J^\mu_{Z'} = \sum_{f=\nu,e,u,d}\left[-\frac{e}{sc}\left(T^3_f - s^2 Q_f\right)s' + e_l g_E c'\right]\bar\psi_f\gamma^\mu \mathbf{P}_{\rm L}\psi_f + \sum_{f=e,u,d}\left[-\frac{e}{sc}\left(-s^2 Q_f\right)s' - e_{f^c} g_E c'\right]\bar\psi_f\gamma^\mu \mathbf{P}_{\rm R}\psi_f, \quad (21)$$

where $\mathbf{P}_{\rm L} = \frac{1}{2}(1-\gamma^5)$, $\mathbf{P}_{\rm R} = \frac{1}{2}(1+\gamma^5)$. The factors in square brackets in (20) and (21) define the couplings $c^{Z^0}_{f{\rm L,R}}$ and $c^{Z'}_{f{\rm L,R}}$.

Gauge invariance of the Yukawa couplings of the matter fields

$$\mathcal{L}_{\rm Yuk} = -y_e H_i^* l_i e^c - y_t \epsilon_{ij} H_i q_j u^c - y_d H_i^* q_i d^c$$

imposes the conditions (see (3))

$$Y^a_{e^c} + Y^a_l - Y^a_H = 0,$$
$$Y^a_{u^c} + Y^a_q + Y^a_H = 0,$$
$$Y^a_{d^c} + Y^a_q - Y^a_H = 0,$$

where $a = E, Y$. When combined with (12) they imply

$$e_{e^c} + e_l - e_H = 0,$$
$$e_{u^c} + e_q + e_H = 0, \quad (22)$$
$$e_{d^c} + e_q - e_H = 0.$$

3 ρ parameters in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model and the Appelquist–Carrazzone decoupling

In this section we define various measurable ρ parameters in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model and show that at the tree level the effects of the heavy Z' decouple. We then identify the dimension six operators which, when added to the SM Lagrangian, reproduce at the tree level the leading (in inverse powers of v_S^2) corrections to the low energy observables due to Z'.

3.1 ρ parameters

In the SM the measurable parameter ρ can be defined in several different ways. The simplest is the definition of ρ (call it $\rho_{\rm low}$) as the ratio of the coefficients of the neutral and charged current terms in the effective low energy four-fermion Lagrangian. Another one is

$$\rho = \frac{M_W^2}{M^2_{Z^0}(1-\sin^2\theta)}, \quad (23)$$

with $\sin^2\theta$ related to measurable quantities in various ways, e.g. as the parameter in the on-shell Z^0 couplings to fermions as in (24), or by the low energy neutral current Lagrangian for e.g. neutrino processes (i.e. as a parameter measuring the admixture of the vector-like electromagnetic current in the leptonic weak neutral current in the low energy four-fermion Lagrangian mentioned above). Finally, ρ (call it ρ_{Zf}) can be defined through the coupling of the on-shell Z^0 to fermion–antifermion pairs expressed in terms of the Fermi constant measured in the muon decay:

$$\mathcal{L}^{Z^0 f\bar f\ \text{on shell}}_{\rm eff} = -\left(\sqrt{2}G_{\rm F}M^2_{Z^0}\rho_{Zf}\right)^{1/2}\bar\psi_f\gamma^\mu \times\left(T^3_f - 2Q_f\sin^2\theta^f_{\rm eff} - T^3_f\gamma^5\right)\psi_f Z^0_\mu. \quad (24)$$

Independently of the definition used, $\rho = 1$ at the tree level due to the custodial $SU(2)_V$ symmetry of the SM Higgs potential. Thus, in the SM $\rho = 1$ is the so-called natural relation, i.e. the prediction which does not depend on the values of the parameters of the model. Of course, quantum corrections to ρ are numerically different for different definitions and do depend on the values of the SM parameters. The usefulness of ρ stems from the fact that the dominant contributions (dependent on the top quark and Higgs boson masses) to it are universal, that is, the same for all definitions of ρ.

Although the different ρ are observables (they are all defined in terms of measurable quantities) none of them can be used as an input observable in the procedure of renormalization of the SM, just because $\rho = 1$ is the natural relation.

In the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model custodial symmetry is broken at the tree level by the Z^0–Z' mixing. It is then necessary to discuss the analogous ρ parameters in some detail. The parameters ρ and ρ_{Zf} can be defined as in the SM, i.e. by (23) and (24), respectively. The parameter $\rho_{\rm low}$ is special, because it refers to the specific form of the low energy effective Lagrangian which needs not be the same as in the SM. In models in which the charged weak currents are unmodified with respect to the SM the effective Lagrangian for low energy weak interactions takes the general form

$$\begin{aligned}\mathcal{L}_{\rm eff} = &-2\sqrt{2}G_{\rm F} J^\mu_+ J_{-\mu} + \frac{1}{2} \\ &\times \sum_{f_1}\sum_{f_2}\Big[a^{f_1f_2}_{\rm LL}\left(\bar\psi_{f_1}\gamma^\mu \mathbf{P}_{\rm L}\psi_{f_1}\right)\left(\bar\psi_{f_2}\gamma^\mu \mathbf{P}_{\rm L}\psi_{f_2}\right) \\ &+ a^{f_1f_2}_{RR}\left(\bar\psi_{f_1}\gamma^\mu \mathbf{P}_{\rm R}\psi_{f_1}\right)\left(\bar\psi_{f_2}\gamma^\mu \mathbf{P}_{\rm R}\psi_{f_2}\right) \\ &+ a^{f_1f_2}_{\rm LR}\left(\bar\psi_{f_1}\gamma^\mu \mathbf{P}_{\rm L}\psi_{f_1}\right)\left(\bar\psi_{f_2}\gamma^\mu \mathbf{P}_{\rm R}\psi_{f_2}\right) \\ &+ a^{f_1f_2}_{\rm RL}\left(\bar\psi_{f_1}\gamma^\mu \mathbf{P}_{\rm R}\psi_{f_1}\right)\left(\bar\psi_{f_2}\gamma^\mu \mathbf{P}_{\rm L}\psi_{f_2}\right)\Big],\end{aligned} \qquad (25)$$

where $J^\mu_\pm$ are the standard charged currents. In the SM the second part of (25) can be rewritten in the form of the product of two neutral currents

$$\mathcal{L}^{\rm NC}_{\rm eff} = -2\sqrt{2}G_{\rm F} J^\mu J_\mu\,, \qquad (26)$$

where

$$J^\mu = \sum_f \sqrt{\rho_f}\,\bar\psi_f\gamma^\mu\left(T^3_f \mathbf{P}_{\rm L} - \sin^2\theta^{\rm eff}_f Q_f\right)\psi_f\,. \qquad (27)$$

Moreover, if the fermion mass effects are neglected ρ_f and $\sin^2\theta^{\rm eff}_f$ are universal, $\rho_f = \rho$, and $\sin^2\theta^{\rm eff}_f = \sin^2\theta^{\rm eff}$. ρ can then be factorized out of the neutral current J^μ, and $\rho = 1$.

The necessary condition to define the low energy parameter ρ_f (possibly dependent on the fermion type) in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model is that the second part of (25) can be written in the current $\times$ current form (26). One would then have

$$\sqrt{\rho_{f_1}\rho_{f_2}} = -\frac{a^{f_1f_2}_{\rm LL} + a^{f_1f_2}_{RR} - a^{f_1f_2}_{\rm LR} - a^{f_1f_2}_{\rm RL}}{\sqrt{2}G_{\rm F}2T^3_{f_1}2T^3_{f_2}}\,. \qquad (28)$$

Computing the diagrams with exchanges of Z^0 and Z' between the two currents $J^\mu_{Z^0}$, see (20), and two currents $J^\mu_{Z'}$, see (21), respectively, and exploiting the relations (A.2) and (A.3), it is easy to find

$$\begin{aligned}&a^{f_1f_2}_{\rm LL} + a^{f_1f_2}_{RR} - a^{f_1f_2}_{\rm LR} - a^{f_1f_2}_{\rm RL} = \\ &-\frac{1}{v_H^2}2T^3_{f_1}2T^3_{f_2} - \frac{\left(\begin{array}{c}\left(2T^3_{f_1}e_H g_E + e_{f_1}g_E + e_{f_1^c}g_E\right)\\ \times\left(2T^3_{f_2}e_H g_E + e_{f_2}g_E + e_{f_2^c}g_E\right)\end{array}\right)}{e_S^2 g_E^2 v_S^2}\end{aligned} \qquad (29)$$

Due to the relations (22) the second term vanishes and, since at the tree level $1/v_H^2 = \sqrt{2}G_{\rm F}$, we find (to some surprise) that in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model at the tree level

$$a^{f_1f_2}_{\rm LL} + a^{f_1f_2}_{RR} - a^{f_1f_2}_{\rm LR} - a^{f_1f_2}_{\rm RL} = -2T^3_{f_1}2T^3_{f_2}\sqrt{2}G_{\rm F} \qquad (30)$$

as in the SM. However, writing the second part of (25) in the familiar current $\times$ current form is not always possible. It is only possible, if the following consistency condition holds:

$$\left(a^{f_1f_2}_{RR} - a^{f_1f_2}_{\rm LR}\right)\left(a^{f_1f_2}_{RR} - a^{f_1f_2}_{\rm RL}\right) = -4\sqrt{2}G_{\rm F}a^{f_1f_2}_{RR} \qquad (31)$$

(it follows from the fact that the form (26) depends only on three unknown: $\sqrt{\rho_{f_1}\rho_{f_2}}$, $\sin^2\theta^{\rm eff}_{f_1}$ and $\sin^2\theta^{\rm eff}_{f_2}$, whereas the general form of the second term in (25) has four independent coefficients). It is straightforward to check that the condition (31) is not satisfied in general. It is satisfied only by that part of (25) which describes neutrino reactions. In this case $a^{f_1\nu_i}_{\rm LR} = a^{f_1\nu_i}_{RR} = a^{\nu_j\nu_i}_{RR} = 0$ and the condition (31) is trivially satisfied. Thus, for neutrino processes one can define the analog of the SM ρ parameter as $\rho_{\rm low} \equiv \sqrt{\rho_\nu\rho_f}$ and from (29) it follows that at the tree level $\rho_{\rm low} = 1$ as in the SM.

In the general case in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model even the generalized low energy parameters ρ_f cannot be defined because the second part of the effective Lagrangian (25) cannot be written in the current $\times$ current form.

It is interesting to contrast $\rho_{\rm low}$ discussed above, for which $\rho_{\rm low} = 1$ at the tree level is a natural relation, with e.g. $\rho = M^2_W/M^2_{Z^0}(1 - \sin^2\theta)$, with $\sin^2\theta$ identified with $\sin^2\theta^\ell_{\rm eff}$ in (24). We find

$$\begin{aligned}\sin^2\theta^\ell_{\rm eff} &= s^2\,\frac{1 - \frac{c}{s}e_{\ell^c}\frac{g_E}{e}\frac{s'}{c'}}{1 - 2sc\,e_H\frac{g_E}{e}\frac{s'}{c'}} \\ &\approx s^2 + s^2\left(2sc\,e_H - \frac{c}{s}e_{\ell^c}\right)\frac{g_E}{e}\frac{s'}{c'} + \dots \\ &= s^2 + \left(s^2\,e_H - \frac{1}{2}e_{\ell^c}\right)\frac{e_H v_H^2}{e_S^2 v_S^2} + \dots\,,\end{aligned} \qquad (32)$$

where we have used (A.1).[3] Using (19) we then get

$$\rho \approx \left(1+\frac{e_H^2 g_E^2 v_H^2}{e_S^2 g_E^2 v_S^2}+\dots\right) \times\left[1+\left(\frac{s^2}{c^2}e_H-\frac{1}{2c^2}e_{\ell^c}\right)\frac{e_H v_H^2}{e_S^2 v_S^2}+\dots\right] = 1+\mathcal{O}\left(\frac{v_H^2}{v_S^2}\right). \quad (33)$$

The important difference between $\rho_{\rm low}$ and ρ in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model is that the latter does depend on some combination of the Lagrangian parameters.[4] From the above results it is clear that the Appelquist–Carrazzone decoupling holds at the tree level in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model. It is also easy to show that it can be easily masked by choosing a low energy observable like $\sin^2\theta$ (and in addition $M_{Z'}$) to fix e.g. the coupling g_E. To simplify the argument, let us assume that $e_{\ell^c}=0$ (at the renormalization scale we are working). Then $e_H^2 v_H^2/e_S^2 v_S^2$ in (33) can be directly expressed in terms of $\sin^2\theta^\ell_{\rm eff}$ from (32) so that

$$\rho \approx \left(\frac{\sin^2\theta^\ell_{\rm eff}}{s^2}+\dots\right)\left[1+\frac{\sin^2\theta^\ell_{\rm eff}-s^2}{c^2}+\dots\right], \quad (34)$$

and the decoupling is lost!

In the next subsection we show the dimension six operators completing the SM Lagrangian, which reproduce leading terms of the corrections to the electroweak observables found at the tree level.

3.2 Decoupling at the tree level

At the tree level the subgroup $U(1)_E$ can be broken independently of the breaking of $SU(2)_{\rm L}\times U(1)_Y$. In this case the gauge field E_μ becomes Z' with a mass $M^2_{Z'}=e_S^2 g_E^2 v_S^2$. For v_S much higher than the Fermi scale, the electroweak observables can be calculated in the $SU(2)_{\rm L}\times U(1)_Y$ effective theory (which is just the SM) supplemented with higher dimensional operators generated by decoupling of heavy Z'. This approach yields corrections to the electroweak observables due to Z' effects in the form of power series in $1/v_S$. Below we display the dimension six operators which reproduce the corrections to different ρ and $\sin^2\theta$ from the preceding subsection up to $\mathcal{O}(1/v_S^4)$.

[3] Defining $\sin^2\theta$ in terms of the structure of the current (27) for neutrino processes we would get

$$\sin^2\theta = s^2+(e_H+e_l)\left(s^2 e_H-\frac{1}{2}e_{e^c}\right)\frac{v_H^2}{e_S^2 v_S^2}.$$

[4] The fact that at the tree level $\rho_{\rm low}=1$ as in the SM makes this observable useless for constraining the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model as the effects of new physics will be always much larger in observables which are modified already at the tree level.

Exchanges of Z' between fermion lines are taken into account by adding to the SM Lagrangian the four-fermion non-renormalizable operators of the type

$$\Delta\mathcal{L}_{\rm SM} = -\frac{1}{e_S^2 g_E^2 v_S^2}e_l^2 g_E^2\left[\bar\psi_{l_A}\gamma^\mu\mathbf{P}_{\rm L}\psi_{l_A}\right]\left[\bar\psi_{l_B}\gamma^\mu\mathbf{P}_{\rm L}\psi_{l_B}\right] -\frac{1}{e_S^2 g_E^2 v_S^2}e_l(-e_{e^c})g_E^2\left[\bar\psi_{e^c_A}\gamma^\mu\mathbf{P}_{\rm R}\psi_{e^c_A}\right] \times\left[\bar\psi_{l_B}\gamma^\mu\mathbf{P}_{\rm L}\psi_{l_B}\right]. \quad (35)$$

The kinetic term of the electroweak Higgs doublet H gives rise, through the first diagram of Fig. 1, to a non-renormalizable term of the form

$$\Delta\mathcal{L}_{\rm SM} = -\frac{1}{2}(2e_H g_E)^2\frac{1}{e_S^2 g_E^2 v_S^2} \times\left[H^\dagger\left(g_2 W^a\,T^a+\frac{1}{2}g_y B\right)H\right]^2. \quad (36)$$

Finally, the second diagram shown in Fig. 1 gives rise to the interaction:

$$\Delta\mathcal{L}_{\rm SM} = \sum_f 2e_f e_H g_E^2\frac{1}{e_S^2 g_E^2 v_S^2} \times\left[H^\dagger\left(g_2 W^a_\mu T^a+\frac{1}{2}g_y B_\mu\right)H\right]\left[\bar f\bar\sigma^\mu f\right]. \quad (37)$$

After the electroweak symmetry breaking, the operator (36) gives the correction to the Z^0 mass squared $\Delta M^2_{Z^0}=-(M^2_{Z^0})_{\rm SM}(e_H^2 v_H^2/e_S^2 v_S^2)$, whereas the operator (37) modifies the Z^0 couplings to the SM fermions:

$$\Delta\mathcal{L}_{\rm SM} = -\sum_f\frac{e}{2sc}\frac{e_f e_H}{e_S^2}\frac{v_H^2}{v_S^2}Z^0_\mu\left[\bar f\bar\sigma^\mu f\right] \approx -\sum_f e_f g_E s' Z^0_\mu\left[\bar f\bar\sigma^\mu f\right];$$

they just correspond to terms $e_f g_E s'$ expanded to order $1/v_S^2$ in the Z^0 couplings; see (20).

At the tree level the three operators (35)–(37) reproduce to order $1/M^2_{Z'}\sim 1/v_S^2$ all corrections to the low energy (compared to v_S) observables due to the extended gauge structure of the model. This is equivalent to the statement that the Appelquist–Carrazone decoupling works for Z' (at least) at the tree level.

We can illustrate this approach by calculating the corrections due to the higher dimensional operators (35)–(37)

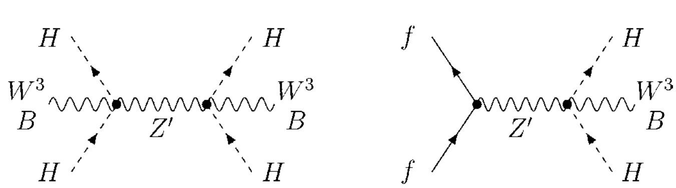

Fig. 1. Generating four-fermion operators by the heavy Z'

to the parameter $\rho_{\rm low}$. To this end it is sufficient to find the difference $a^{\ell\nu}_{\rm LL} - a^{\ell\nu}_{\rm RL}$ of the coefficients in the effective Lagrangian (25). In the SM $a^{\ell\nu}_{\rm LL} - a^{\ell\nu}_{\rm RL} = (e^2/4s^2c^2M^2_{Z^0}) = 1/v^2_H$, and since at the tree level $1/v^2_H = \sqrt{2}G_{\rm F}$, we have $\rho_{\rm low} = 1$. The corrections due to the extended gauge structure read

$$\left(\Delta a^{\ell\nu}_{\rm LL}\right)_{Z'} = -\frac{1}{g^2_E e^2_S v^2_S} e^2_l g^2_E\,,$$
$$\left(\Delta a^{\ell\nu}_{\rm RL}\right)_{Z'} = \frac{1}{g^2_E e^2_S v^2_S} e_l e_{e^c} g^2_E \tag{38}$$

from the operator (35),

$$\left(\Delta a^{\ell\nu}_{\rm LL}\right)_{Z'} = \frac{e^2}{4s^2c^2}\left(1-2s^2\right)\frac{1}{M^2_{Z^0}}\frac{e^2_H v^2_H}{e^2_S v^2_S}\,,$$
$$\left(\Delta a^{\ell\nu}_{\rm RL}\right)_{Z'} = \frac{e^2}{4s^2c^2}\left(-2s^2\right)\frac{1}{M^2_{Z^0}}\frac{e^2_H v^2_H}{e^2_S v^2_S}\,, \tag{39}$$

from the correction to the Z^0 mass produced by the operator (36), and

$$\left(\Delta a^{\ell\nu}_{\rm LL}\right)_{Z'} = -\frac{e^2}{2c^2}\frac{1}{M^2_{Z^0}}\frac{e_l e_H v^2_H}{e^2_S v^2_S}\,,$$
$$\left(\Delta a^{\ell\nu}_{\rm RL}\right)_{Z'} = -\frac{e^2}{4s^2c^2}\frac{1}{M^2_{Z^0}}\left(2s^2 e_l - e_{e^c}\right)\frac{e_H v^2_H}{e^2_S v^2_S}\,, \tag{40}$$

from the correction to the Z^0 couplings produced by the operator (37). Combining these three corrections we find, using the relations (22), that $\Delta(a^{\ell\nu}_{\rm LL} - a^{\ell\nu}_{\rm RL}) = 0$. Other observables can be checked similarly. Corrections subleading in $1/v_S$ can also be reproduced upon inclusion in the SM Lagrangian operators of dimension higher than six.

The equivalence of the two approaches (full calculation versus higher dimensional operators) checked above shows that the Appelquist–Carrazzone decoupling holds at the tree level. The expectation that it should hold in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model to all orders is based on the observation that $U(1)_E$ can be broken independently of the breaking of $SU(2)_{\rm L}\times U(1)_Y$. We will propose the scheme which makes it explicit at one loop and thus show that in particular it is not spoiled by the mixing of the gauge fields corresponding to the two $U(1)$ groups.

4 Renormalization scheme

Before we define our renormalization scheme for the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ extension of the SM, it is instructive to recall the simplest possible approach to calculating loop corrections to the electroweak observables within the SM [15, 16].

Basic (running) parameters of the SM are[5] $\hat{g}_y$, $\hat{g}_2$ and $\hat{v}_H$ (or any three other functions of these parameters, e.g. $\hat{\alpha}$, $\hat{M}_Z$ and $\hat{s}^2$). In the renormalization procedure they are expressed in terms of the values of the three experimentally measured observables. Traditionally one chooses for this purpose $G_{\rm F}$, $\alpha_{\rm EM}$ and M_Z. These quantities are computed in perturbation calculus using for example the dimensional regularization and the $\overline{\rm MS}$ subtraction:

$$\alpha_{\rm EM} = \frac{\hat{g}^2_y\hat{g}^2_2}{4\pi\left(\hat{g}^2_y+\hat{g}^2_2\right)} + \delta\alpha_{\rm EM} = \frac{\hat{e}^2}{4\pi} + \delta\alpha_{\rm EM} = \hat{\alpha} + \delta\alpha_{\rm EM}$$
$$M^2_Z = \frac{1}{4}\left(\hat{g}^2_y+\hat{g}^2_2\right)\hat{v}^2 = \frac{1}{4}\frac{\hat{e}^2}{\hat{s}^2\hat{c}^2}\hat{v}^2 + \delta M^2_Z = \hat{M}^2_Z + \delta M^2_Z\,, \tag{41}$$
$$G_{\rm F} = \frac{1}{\sqrt{2}\hat{v}^2} + \delta G_{\rm F} = \frac{\hat{e}^2}{\sqrt{2}4\hat{s}^2\hat{c}^2\hat{M}^2_Z} + \delta G_{\rm F} = \hat{G}_{\rm F} + \delta G_{\rm F}\,.$$

As the corrections $\delta\alpha_{\rm EM}$, δM^2_Z and $\delta G_{\rm F}$ are calculated in terms of the parameters $\hat{\alpha}$, $\hat{M}^2_Z$ and $\hat{s}^2$ the above relations have to be inverted recursively. At the one-loop order this is particularly simple:

$$\hat{\alpha} = \alpha_{\rm EM} - \delta\alpha_{\rm EM}\,,$$
$$\hat{M}^2_Z = M^2_Z - \delta M^2_Z\,, \tag{42}$$
$$\hat{G}_{\rm F} = G_{\rm F} - \delta G_{\rm F}\,,$$

where in $\delta\alpha_{\rm EM}$, δM^2_Z and $\delta G_{\rm F}$ one replaces the parameters $\hat{\alpha}$, $\hat{M}^2_Z$ and $\hat{s}^2$ by $\alpha_{\rm EM}$, M_Z and $G_{\rm F}$ using the tree-level relations. For any other measurable quantity $\mathcal{A}$ we then have

$$\mathcal{A} = \mathcal{A}^{(0)}\left(\hat{\alpha}, \hat{M}^2_Z, \hat{G}_{\rm F}\right) + \delta\mathcal{A}\left(\hat{\alpha}, \hat{M}^2_Z, \hat{G}_{\rm F}\right) + \dots\,, \tag{43}$$

where $\delta\mathcal{A}$ is the one-loop contribution to the quantity $\mathcal{A}$. This is next written as

$$\mathcal{A} = \mathcal{A}^{(0)}\left(\alpha_{\rm EM}, M^2_Z, G_{\rm F}\right) + \delta\mathcal{A}\left(\alpha_{\rm EM}, M^2_Z, G_{\rm F}\right) - \frac{\partial\mathcal{A}^{(0)}}{\partial\alpha_{\rm EM}}\delta\alpha_{\rm EM} - \frac{\partial\mathcal{A}^{(0)}}{\partial M^2_Z}\delta M^2_Z - \frac{\partial\mathcal{A}^{(0)}}{\partial G_{\rm F}}\delta G_{\rm F}\,. \tag{44}$$

The expression (44) is finite and independent of the renormalization scale μ.

The free running parameters of the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ extension of the SM are g_2, v_H and v_S and the couplings g_{EE}, g_{EY}, g_{YY} and g_{YE} (in fact only three of them). One way of organizing higher loop calculations in such a model is to follow the recipe sketched above and to choose the appropriate number of input observables, in terms of which one would express all the running parameters.

Clearly, for $M_{Z'} \gg M_{Z^0}$ the parameters of the model form two sets: g_2, g_y and v_H describe the SM electroweak sector, and v_S and the remaining gauge couplings describe the Z' sector. However, since the Z' boson has not yet been discovered and its mass is unknown (assuming it exists), the best way to organize loop calculations is such that the Appelquist–Carrazzone decoupling (in the case Z' is heavy) would be manifest. This condition is not satisfied by schemes in which additional parameters related to the heavy particle sector are expressed in terms of low energy

[5] We denote running parameters which are traded for observables by a hat.

observables. Decoupling would be manifest if all additional parameters were related to the measurable characteristics of the heavy particles. Independently of the question of decoupling, renormalization schemes using a number of observables equal to the number of free parameters may be difficult to implement in practice as one has to solve for the running parameters a larger set of equations than (41) in the SM, and the resulting analytical formulae may be very complicated and unmanageable.

In the fits to the electroweak data, breakdown of explicit Appelquist–Carrazzone decoupling in a scheme chosen to compute the observables may even incorrectly produce upper bounds on the additional heavy particles (gauge bosons, Higgs scalars).

In this paper we propose to organize loop calculations into a hybrid scheme in which the parameters $\hat{g}_2$, $\hat{g}_y$ and $\hat{v}_H$ are expressed in terms of $\alpha_{\rm EM}$, $G_{\rm F}$ and M_{Z^0} (or M_W) as in the SM and the remaining parameters are kept in the calculations as the $\overline{\rm MS}$ scheme running parameters. The renormalization scale μ for them can be chosen arbitrarily.

As we will show by explicit calculations in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model, the advantage of such a hybrid scheme[6] is twofold: the Appelquist–Carrazzone decoupling of heavy particle effects is made manifest – for heavy particle masses taken to infinity the expressions for the observables measured at energies of the order of the electroweak scale (or lower) coincide with the SM expression due to the presence of explicit suppression by a large mass scale (in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model by factors of $1/v_S^2$). Moreover, an explicit renormalization scale dependence remains only in the terms suppressed by the large mass scale(s). The expressions for observables are in fact scale independent when the RG running of the parameters is taken into account. Tadpoles play a crucial role here [17]. Last but not least, our scheme does not require solving for running parameters a complicated set of equations; in this respect it is as practical in use as the usual schemes in the SM.

Extensions of the SM are constrained by precision electroweak observables. In our scheme observables are calculated in terms of $\alpha_{\rm EM}$, $G_{\rm F}$ and M_Z or M_W (because in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model the tree-level formula (19) for the Z^0 mass is complicated it is much more convenient to take as the three input observables $\alpha_{\rm EM}$, $G_{\rm F}$ and M_W^2 and compute instead $M^2_{Z^0}$ in terms of these) and the additional parameters of the model at a conveniently chosen renormalization scale μ. Fits to the data can then give constraints on these running parameters. Moreover, in theories in which the Appelquist–Carrazzone decoupling holds, because the loop corrections reduce to their SM form as the heavy mass scale is sent to infinity, a fairly accurate estimate of the limits imposed by the precision data on the additional parameters of the model is possible by combining the SM loop corrections with the tree-level corrections due to "new physics".

The one-loop expressions for the chosen basic input observables read (see Appendix B for details):

$$\begin{aligned}\hat{\alpha} &= \frac{\alpha_{\rm EM}}{1+\hat{\hat{\Pi}}_\gamma(0)-(\hat{\alpha}/\pi)\ln\frac{\hat{M}_W^2}{\mu^2}}\\ &\approx \alpha_{\rm EM}\left(1-\hat{\hat{\Pi}}_\gamma(0)+\frac{\alpha_{\rm EM}}{\pi}\ln\frac{M_W^2}{\mu^2}\right)\\ \hat{M}_W^2 &= M_W^2\left(1-\frac{\hat{\Pi}_{WW}(M_W^2)}{M_W^2}\right)\\ \hat{v}_H^2 &= \frac{1}{\sqrt{2}G_{\rm F}}(1+\Delta_G)\,,\end{aligned} \tag{45}$$

with Δ_G given in (B.4) and

$$\begin{aligned}\hat{s}^2 &= \frac{\pi\alpha_{\rm EM}}{\sqrt{2}G_{\rm F}M_W^2}(1+\Delta)\equiv s^2_{(0)}+s^2_{(0)}\Delta\,,\\ \hat{c}^2 &= \frac{\sqrt{2}G_{\rm F}M_W^2-\pi\alpha_{\rm EM}(1+\Delta)}{\sqrt{2}G_{\rm F}M_W^2}\equiv c^2_{(0)}-s^2_{(0)}\Delta\,,\end{aligned} \tag{46}$$

where

$$\Delta = -\hat{\hat{\Pi}}_\gamma(0)+\frac{\hat{\alpha}}{\pi}\ln\frac{\hat{M}_W^2}{\mu^2}+\frac{\hat{\Pi}_{WW}(M_W^2)}{M_W^2}+\Delta_G \tag{47}$$

(as usual $\hat{\hat{\Pi}}_\gamma(q^2)$ is defined by $\hat{\Pi}_{\gamma\gamma}(q^2)=q^2\hat{\hat{\Pi}}_\gamma(q^2)$, i.e. it is the residue of the photon propagator).

Using this scheme we will explicitly demonstrate that in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ extension of the SM the Appelquist–Carrazzone decoupling does hold. To this end we will compute in our scheme the two different ρ parameters defined as in Sect. 3 in terms of the following observables: $\rho_{\rm low}$, defined by the effective Lagrangian for $\nu_\mu e^-$ elastic scattering, and $\rho\equiv M_W^2/M_Z^2(1-\sin^2\theta^\ell_{\rm eff})$, where $\sin^2\theta^\ell_{\rm eff}$ parametrizes the effective coupling of an on-shell Z^0 to an l^+l^- pair. In particular we will demonstrate that the celebrated m_t^2/M_W^2 term is present in both cases.

5 Decoupling of Z' effects in $\rho_{\rm low}$ at one loop

As an exercise, in order to demonstrate the working of our renormalization scheme, we will compute one-loop corrections to the low energy parameter $\rho_{\rm low}$ defined by the $\nu_\mu e^-\to\nu_\mu e^-$ elastic scattering. Since $\rho_{\rm low}=1$ at the tree level is a natural relation in the $SU(2)\times U(1)_Y\times U(1)_E$ model, the one-loop corrections to $\rho_{\rm low}$ should be finite when $1/v_H^2$ in (29) is expressed in terms of $G_{\rm F}$ with one-loop accuracy.

At one loop the direct generation number dependent fermion contribution comes through the "oblique" correc-

[6] In fact, such a hybrid scheme is adopted for the usual treatment of the strong interaction corrections to the electroweak observables: $\hat{\alpha}_s(\mu)$ is not traded for any observable quantity; instead one relies on the fact that the explicit μ dependence of the two-loop contributions should cancel against the μ dependence of $\hat{\alpha}_s(\mu)$ in one-loop terms.

tions to $a^{e\nu}_{\rm LL} - a^{e\nu}_{\rm RL}$:

$$\begin{aligned}(a^{e\nu}_{\rm LL} - a^{e\nu}_{\rm RL})_{\text{1-loop}} &= c^{Z^0}_{\nu \rm L} a^{Z^0}_e \frac{1}{M^2_{Z^0}} \frac{\Pi_{Z^0Z^0}(0)}{M^2_{Z^0}} \\ &+ c^{Z'}_{\nu \rm L} a^{Z'}_e \frac{1}{M^2_{Z'}} \frac{\Pi_{Z'Z'}(0)}{M^2_{Z'}} \\ &+ c^{Z'}_{\nu \rm L} a^{Z^0}_e \frac{1}{M^2_{Z^0}} \frac{\Pi_{Z^0Z'}(0)}{M^2_{Z'}} \\ &+ c^{Z^0}_{\nu \rm L} a^{Z'}_e \frac{1}{M^2_{Z^0}} \frac{\Pi_{Z^0Z'}(0)}{M^2_{Z'}} , \quad (48)\end{aligned}$$

where Z_i denotes Z^0 or Z', $a^{Z_i}_f = c^{Z_i}_{f\rm L} - c^{Z_i}_{f\rm R}$, and the couplings $c^{Z^0}_{f\rm L,R}$, $(c^{Z'}_{f\rm L,R})$ of Z^0 (Z') to left- and right-chiral leptons are defined by (20) and (21). The self-energies $\Pi_{Z_iZ_j}$ contain in principle also tadpole contributions. Another generation-number dependent contribution to ρ arises from $\hat\Pi_{WW}(0)/\hat M^2_W$ after expressing $1/\hat v^2_H$ in the tree-level term (29) with one-loop accuracy

$$(a^{e\nu}_{\rm LL} - a^{e\nu}_{\rm RL})_{\rm tree} = \frac{1}{\hat v^2_H} = \sqrt{2} G_{\rm F}(1 - \Delta_G) , \quad (49)$$

with Δ_G given by (B.4).

Fermionic contribution to $\rho_{\rm low}$

The top–bottom quark contribution to the one-particle irreducible part of $\hat\Pi_{WW}$ is the same as in the SM:

$$\begin{aligned}\hat\Pi_{WW}(0) = \frac{\hat e^2}{\hat s^2} N_c \Big[& 2\tilde A(0, m_t, m_b) - \frac{1}{2}(m^2_t + m^2_b) \\ & \times b_0(0, m_t, m_b) \Big] , \quad (50)\end{aligned}$$

where $N_c = 3$. The one-particle irreducible part of $\hat\Pi_{Z_iZ_j}(0)$ can be simplified to

$$\begin{aligned}\hat\Pi_{Z_iZ_j}(0) = &-2a^{Z_i}_t a^{Z_j}_t N_c m^2_t b_0(0, m_t, m_t) \\ &- 2a^{Z_i}_b a^{Z_j}_b N_c m^2_b b_0(0, m_b, m_b) .\end{aligned}$$

Contributions of the other fermions can be written analogously. When inserted into (48) the fermion f contribution to $\hat\Pi_{Z_iZ_j}(0)$ factorizes as

$$\begin{aligned}(a^{e\nu}_{\rm LL} - a^{e\nu}_{\rm RL})^{(f)}_{\text{1-loop}} = &-\left(\frac{a^{Z^0}_e a^{Z^0}_f}{M^2_{Z^0}} + \frac{a^{Z'}_e a^{Z'}_f}{M^2_{Z'}} \right) \\ &\times \left(\frac{c^{Z^0}_{\nu \rm L} a^{Z^0}_f}{M^2_{Z^0}} + \frac{c^{Z'}_{\nu \rm L} a^{Z'}_f}{M^2_{Z'}} \right) \\ &\times 2m^2_f N_c\, b_0(0, m_f, m_f) ,\end{aligned}$$

and computing the factors in brackets using (20) and (21) and the formulae (A.2) and (A.3) one finds (omitting $1/16\pi^2$)

$$\begin{aligned}(a^{e\nu}_{\rm LL} - a^{e\nu}_{\rm RL})^{t,b}_{\text{1-loop}} &= \frac{2}{v^4_H} m^2_t N_c \ln \frac{m^2_t}{\mu^2} \\ &\times \left[1 - \frac{v^2_H}{v^2_S} \frac{(e_l + e_{e^c} - e_H)(e_q + e_{u^c} + e_H)}{e^2_S} \right] \\ &\times \left[1 + \frac{v^2_H}{v^2_S} \frac{(e_l + e_H)(e_q + e_{u^c} + e_H)}{e^2_S} \right] \\ &+ \frac{2}{v^4_H} m^2_b N_c \ln \frac{m^2_b}{\mu^2} \\ &\times \left[1 + \frac{v^2_H}{v^2_S} \frac{(e_l + e_{e^c} - e_H)(e_q + e_{d^c} - e_H)}{e^2_S} \right] \\ &\times \left[1 - \frac{v^2_H}{v^2_S} \frac{(e_l + e_H)(e_q + e_{d^c} - e_H)}{e^2_S} \right] .\end{aligned}$$

The first terms in square brackets reproduce the SM contribution. The other terms are simply zero due to the relations (22). Combining this with the top–bottom contribution to $\hat\Pi_{WW}(0)$ in (49) one finds that the fermionic "oblique" contribution to $\rho_{\rm low}$ is finite and exactly reproduces the one-loop SM result

$$\Delta\rho_{\rm low} = \frac{N_c}{16\pi^2} \sqrt{2} G_{\rm F} g(m_t, m_b) + \ldots = \frac{N_c}{16\pi^2} \sqrt{2} G_{\rm F} m^2_t + \ldots \quad (51)$$

(the function $g(m_1, m_2)$ is defined in Appendix E). Thus, we explicitly demonstrate that in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model the celebrated $\propto m^2_t$ contribution is present in the ρ parameter defined in terms of low energy neutrino processes.

Bosonic contribution $\rho_{\rm low}$

The circumstance simplifying calculation of the vertex and self-energy corrections to external lines to the $\nu_\mu e^- \to \nu_\mu e^-$ amplitude is that (due to the corresponding $U(1)$ Ward identities) the corrections to the vertices due to the virtual Z^0 and Z' are exactly canceled by the virtual Z^0 and Z' contributions to the self-energies. For the corrections due to the virtual W one finds

$$\begin{aligned}16\pi^2\, (a^{e\nu}_{\rm LL})^{\rm vert}_{\text{1-loop}} = \Bigg[& c^{Z^0}_{e\rm L} \frac{1}{M^2_{Z^0}} \left(\hat e^3 \frac{\hat c}{\hat s^3} c' \right) \\ & + c^{Z'}_{e\rm L} \frac{1}{M^2_{Z'}} \left(-\hat e^3 \frac{\hat c}{\hat s^3} s' \right) \\ & + c^{Z^0}_{\nu \rm L} \frac{1}{M^2_{Z^0}} \left(-\hat e^3 \frac{\hat c}{\hat s^3} c' \right) + c^{Z'}_{\nu \rm L} \frac{1}{M^2_{Z'}} \\ & \times \left(\hat e^3 \frac{\hat c}{\hat s^3} s' \right) \Bigg] \left(\eta_{\rm div} + \ln \frac{\hat M^2_W}{\mu^2} \right) , \quad (52)\end{aligned}$$

$$16\pi^2\,(a^{e\nu}_{\rm RL})^{\rm vert}_{\text{1-loop}} = \left[c^{Z^0}_{e{\rm R}}\frac{1}{M^2_{Z^0}}\left(\hat{e}^3\frac{\hat{c}}{\hat{s}^3}c'\right) + c^{Z'}_{e{\rm R}}\frac{1}{M^2_{Z'}}\left(-{\rm i}\hat{e}^3\frac{\hat{c}}{\hat{s}^3}s'\right)\right] \times\left(\eta_{\rm div}+\ln\frac{\hat{M}^2_W}{\mu^2}\right), \quad (53)$$

and, after using the relations (A.2) and (A.3),

$$16\pi^2\,(a^{e\nu}_{\rm LL}-a^{e\nu}_{\rm RL})^{\rm vert}_{\text{1- loop}} = -\frac{4}{\hat{v}^2_H}\hat{e}^2\frac{\hat{c}^2}{\hat{s}^2} \times\left[1+\frac{1}{2}\frac{\hat{v}^2_H}{\hat{v}^2_S}\frac{2e^2_H-e_He_{e^c}}{e^2_S}\right]\left(\eta_{\rm div}+\ln\frac{\hat{M}^2_W}{\mu^2}\right). \quad (54)$$

Using (A.2), (A.3) and the results for $\hat{\Pi}_{\gamma Z^0}(0)$ and $\hat{\Pi}_{\gamma Z'}(0)$ which can be extracted from Appendix B.1, one can also check that the "oblique" corrections to the $\nu_\mu e\to\nu_\mu e$ scattering amplitude potentially singular at zero momentum transfer cancel against the singular contribution of the photon exchange between the tree level $ee\gamma$ and one-loop $\nu\nu\gamma$ vertices as in the SM [16].

The bosonic contribution to (48) can be calculated using the formulae collected in Appendix D. The structure of the W^+W^-, $G^\pm W^\mp$, G^+G^-, G^0h^0 and $G'S^0$ contribution to $\Pi_{Z_iZ_j}$ is such that they can be written in the form

$$\Pi^{(k)}_{Z_iZ_j}(q^2)=\alpha^{(k)}_{Z_i}\alpha^{(k)}_{Z_j}\Pi^{(k)}(q^2)\,, \quad (55)$$

which, when used in the $e\nu\to e\nu$ amplitude, leads to the factorization observed already for the fermionic contribution:

$$a^{e\nu}_{\rm LL}=\left(\frac{c^{Z^0}_{\nu{\rm L}}\alpha^{(k)}_{Z^0}}{M^2_{Z^0}}+\frac{c^{Z'}_{\nu{\rm L}}\alpha^{(k)}_{Z'}}{M^2_{Z'}}\right) \times\left(\frac{c^{Z^0}_{e{\rm L}}\alpha^{(k)}_{Z^0}}{M^2_{Z^0}}+\frac{c^{Z'}_{e{\rm L}}\alpha^{(k)}_{Z'}}{M^2_{Z'}}\right)\Pi^{(k)}(q^2)\,, \quad (56)$$

and similarly for $a^{e\nu}_{\rm RL}$. This allows one to easily calculate the divergent part of the corresponding contributions to $a^{e\nu}_{\rm LL}-a^{e\nu}_{\rm RL}$ (of these only W^+W^- and $G^\pm W^\mp$ are divergent). Using the tricks (A.2), (A.3) and (22) it is

$$\frac{1}{\hat{v}^2_H}2\hat{e}^2\frac{\hat{c}^4}{\hat{s}^2}\left[1+\frac{v^2_H}{v^2_S}\frac{e_H(e_H+e_l)}{e^2_S}\right] -\frac{\hat{e}^2}{\hat{v}^2_H}\left(2\hat{s}^2-2\hat{c}^2\frac{v^2_H}{v^2_S}\frac{e_H(e_H+e_l)}{e^2_S}\right)\eta_{\rm div}\,. \quad (57)$$

The divergences of the Z^0h^0 and $Z'h^0$ loop contributions to $\Pi_{Z_iZ_j}$ can be combined to yield

$$\left[\Pi_{Z_iZ_j}\right]_{\rm div}=\alpha_{Z_i}\alpha_{Z_j}\left(\frac{\hat{e}^2}{\hat{s}^2\hat{c}^2}+4e^2_Hg^2_E\right)\frac{\hat{v}^2_H}{4}\eta_{\rm div}\,,$$

with $\alpha_{Z^0}=-\frac{\hat{e}}{\hat{s}\hat{c}}c'+2e_Hg_Es'$ and $\alpha_{Z'}=\frac{\hat{e}}{\hat{s}\hat{c}}c'+2e_Hg_Es'$. The corresponding divergent contributions to $a^{e\nu}_{\rm LL}-a^{e\nu}_{\rm RL}$ is then

$$-\frac{1}{\hat{v}^2_H}\left(\frac{\hat{e}^2}{\hat{s}^2\hat{c}^2}+4e^2_Hg^2_E\right)\eta_{\rm div}\,. \quad (58)$$

The other "oblique" bosonic contributions are finite. It is also easy to check that the tadpole contributions to the vector boson self-energies cancel in the difference $a^{e\nu}_{\rm LL}-a^{e\nu}_{\rm RL}$.

Finally we record for completeness the finite contributions of the box diagrams to the coefficients $a^{\nu e}_{\rm LL}$ and $a^{\nu e}_{\rm LR}$ of the low energy Lagrangian (25). We find

$$16\pi^2a^{\nu e}_{\rm LL}=\frac{1}{M^2_{Z^0}}3\left(c^{Z^0}_{\nu{\rm L}}\right)^2\left(c^{Z^0}_{e{\rm L}}\right)^2+\frac{1}{M^2_{Z'}}3\left(c^{Z'}_{\nu L}\right)^2\left(c^{Z'}_{e{\rm L}}\right)^2 +\frac{1}{M^2_{Z'}-M^2_{Z^0}}\ln\left(\frac{M^2_{Z'}}{M^2_{Z^0}}\right)6\,c^{Z^0}_{\nu{\rm L}}c^{Z'}_{\nu{\rm L}}c^{Z^0}_{e{\rm L}}c^{Z'}_{e{\rm L}} +\frac{\hat{e}^4}{\hat{s}^4M^2_W}\,, \quad (59)$$

$$16\pi^2a^{\nu e}_{\rm LR}=-\frac{1}{M^2_{Z^0}}3\left(c^{Z^0}_{\nu{\rm L}}\right)^2\left(c^{Z^0}_{e{\rm R}}\right)^2-\frac{1}{M^2_{Z'}}3\left(c^{Z'}_{\nu{\rm L}}\right)^2\left(c^{Z'}_{e{\rm R}}\right)^2 -\frac{1}{M^2_{Z'}-M^2_{Z^0}}\ln\left(\frac{M^2_{Z'}}{M^2_{Z^0}}\right)6c^{Z^0}_{\nu{\rm L}}c^{Z'}_{\nu{\rm L}}c^{Z^0}_{e{\rm R}}c^{Z'}_{e{\rm R}}\,.$$

From these formulae the box contribution to $\rho_{\rm low}$ can be easily obtained.

Combining the results (54), (57) and (58) with the divergent part of Δ_G in (49) given by (B.5) and (D.2), one easily finds that the total one-loop contribution to the $\rho_{\rm low}$ parameter defined in terms of the $\nu e\to\nu e$ scattering amplitude is finite and, since the coefficient of $\ln(1/\mu^2)$ is the same as that of $\eta_{\rm div}$, is independent of the renormalization scale. Moreover, it is easy to see that in the limit $v_S\to\infty$ one recovers the SM result, i.e. the Appelquist–Carrazzone decoupling is manifest.

If $\sin^2\theta^{\rm eff}_\ell$ is used as an additional observable, the explicit decoupling is lost. This is because one has then to express g_E and v_S in the one-loop contribution through $M_{Z'}$ and $\sin^2\theta^{\rm eff}_\ell$ (to zeroth-order accuracy) with the effect already described: the explicit suppression factor $\propto 1/v^2_S$ is then replaced by the difference of $\sin^2\theta^{\rm eff}_\ell-s^2_{(0)}$ which is finite and does not vanish as $v_S\to\infty$.

6 One-loop calculation of $M^2_{Z^0}$

In this section we compute $M^2_{Z^0}$ in our scheme. Unlike the previous example of $\rho_{\rm low}$, the tree-level formula for $M^2_{Z^0}$ does depend on the parameters of the extended gauge sector. Therefore, in the one-loop result for $M^2_{Z^0}$ in our scheme, the explicit dependence on the renormalization scale μ will remain. We will however show that the conditions for the heavy Z' effects to decouple are satisfied: the part of the result which does not vanish as $v_S\to\infty$ is independent of μ and takes the SM form. Furthermore, we will show that the whole result for $M^2_{Z^0}$ is independent of the renormalization scale if the dependence on μ of the parameters in the zeroth-order expression is taken into account. This constitutes a non-trivial check of the renormalization group equations (13)–(15) and of our renormalization scheme.

We calculate now the one-loop corrections to the Z^0 boson mass. It is given by the formula[7]

$$M^2_{Z^0} = \hat{M}^2_{Z^0} + \Pi_{Z^0Z^0}\left(M^2_{Z^0}\right) ,$$

where the tree-level term $\hat{M}^2_{Z^0}$ is given by (19). The running parameters $\hat{e}$, $\hat{s}$, $\hat{c}$ and $\hat{v}_H$ in $\hat{M}^2_{Z^0}$ have to be expressed in terms of the input observables $G_{\rm F}$, M^2_W and $\alpha_{\rm EM}$ with one-loop accuracy by using the relations (45) and (46). This gives

$$\begin{aligned} A_0+\delta A &= \frac{M^2_W}{c^2_{(0)}}\left\{1-\frac{\hat{\Pi}_{WW}(M^2_W)}{M^2_W}+\frac{s^2_{(0)}}{c^2_{(0)}}\Delta\right\} ,\\ B_0+\delta B &= g^2_E e^2_S v^2_S + \frac{g^2_E e^2_H}{\sqrt{2}G_{\rm F}}\left(1+\Delta_G\right) , \qquad (60)\\ D_0+\delta D &= -\frac{1}{2}e_H g_E \frac{e_{(0)}}{s_{(0)}c_{(0)}\sqrt{2}G_{\rm F}}\\ &\quad\times\left\{1+\frac{1}{2}\frac{s^2_{(0)}}{c^2_{(0)}}\Delta-\frac{1}{2}\frac{\hat{\Pi}_{WW}(M^2_W)}{M^2_W}+\frac{1}{2}\Delta_G\right\} , \end{aligned}$$

where $e_{(0)} \equiv \sqrt{4\pi\alpha_{\rm EM}}$ and Δ and Δ_G are given by (47) and (B.4), respectively. In agreement with the prescription (44) we then have $2M^2_{Z^0} = 2(M^2_{Z^0})_{(0)} + 2\delta M^2_{Z^0}$, where $(M^2_{Z^0})_{(0)}$ is given by (19) with A, B and D replaced by A_0, B_0 and D_0, respectively, and

$$\begin{aligned} 2\delta M^2_{Z^0} &= \delta A+\delta B-\frac{(A_0-B_0)(\delta A-\delta B)+4D_0\delta D}{\sqrt{(A_0-B_0)^2+4D_0^2}}\\ &\quad+2\,\hat{\Pi}_{Z^0Z^0}(M^2_Z)\\ &= \frac{M^2_W}{c^2_{(0)}}\left[-\frac{\hat{\Pi}_{WW}(M^2_W)}{M^2_W}+\frac{s^2_{(0)}}{c^2_{(0)}}\Delta\right]+2\hat{\Pi}_{Z^0Z^0}(M^2_Z)\\ &\quad+\frac{g^2_E e^2_S v^2_S+\frac{g^2_E e^2_H}{\sqrt{2}G_{\rm F}}-\frac{M^2_W}{c^2_{(0)}}}{\sqrt{\cdots}}\\ &\quad\times\left\{\frac{M^2_W}{c^2_{(0)}}\left[-\frac{\hat{\Pi}_{WW}(M^2_W)}{M^2_W}+\frac{s^2_{(0)}}{c^2_{(0)}}\Delta\right]-\frac{g^2_E e^2_H}{\sqrt{2}G_{\rm F}}\Delta_G\right\}\\ &\quad+\frac{g^2_E e^2_H}{\sqrt{2}G_{\rm F}}\Delta_G-\frac{g^2_E e^2_H}{\sqrt{\cdots}}\,\frac{e^2_{(0)}}{2G^2_{\rm F}s^2_{(0)}c^2_{(0)}}\\ &\quad\times\left\{\frac{1}{2}\frac{s^2_{(0)}}{c^2_{(0)}}\Delta-\frac{1}{2}\frac{\hat{\Pi}_{WW}(M^2_W)}{M^2_W}+\frac{1}{2}\Delta_G\right\} , \qquad (61) \end{aligned}$$

where the self-energies $\hat{\Pi}_{WW}$ and $\hat{\Pi}_{Z^0Z^0}$ include the tadpole contributions. We would like now to demonstrate that *i*) in the limit $v_S \to \infty$ the SM result is recovered, and *ii*) that the above result is independent of the renormalization scale μ.

[7] Mixing of Z^0 with Z' is formally a two-loop effect.

6.1 SM limit and decoupling of the heavy Z' effects

For $v_S \to \infty$ the tree-level term $(M^2_{Z^0})_{(0)}$ obviously gives the SM result $M^2_W/c^2_{(0)}$. Moreover, the prefactor in the third line of (61) is then $1+\mathcal{O}(1/v^4_S)$ and the prefactor of the last term is also suppressed by $1/v^2_S$. Thus in the limit one recovers superficially the SM formula. We have

$$\begin{aligned} 2\delta M^2_{Z^0} &\to 2\frac{M^2_W}{c^2_{(0)}}\left[-\frac{\hat{\Pi}_{WW}(M^2_W)}{M^2_W}+\frac{s^2_{(0)}}{c^2_{(0)}}\Delta\right]\\ &\quad+2\hat{\Pi}_{Z^0Z^0}(M^2_Z) . \qquad (62) \end{aligned}$$

However, one still has to check that the appropriate combinations of $\hat{\Pi}_{WW}$, $\hat{\Pi}_{Z^0Z^0}$ and Δ do not contain terms which would grow too fast as $v_S \to \infty$, invalidating the argument.

In order to show that they do not, we first note that the S^0 tadpole $\mathcal{T}_{S^0}$ which contributes only to $\hat{\Pi}_{Z^0Z^0}$ is suppressed (as we show below, the h^0 tadpoles cancel exactly in the full formula (61), similarly as in the SM). Indeed, the S^0 coupling to Z^0Z^0 is proportional to $s'^2 v_S \sim 1/v^3_S$; the S^0 propagator is $\sim 1/v^2_S$; the S^0 coupling to $Z'Z'$ and S^0S^0 pairs is proportional to v_S, so that these particles circulating in the tadpole loop give to $\mathcal{T}_{S^0}$ contributions $\sim v^3_S$. Hence, the S^0 tadpole contribution to $\hat{\Pi}_{Z^0Z^0}$ goes as $\sim (1/v^3_S)(1/v^2_S)(v^3_S) \sim 1/v^2_S$.

Furthermore, Δ approaches in this limit its SM form due to cancellation of the leading terms for $v_S \to \infty$ between Λ and $\Sigma_{\nu{\rm L}}+\Sigma_{e{\rm L}}$ and between $\hat{\Pi}_{WW}(M^2_W)$ and $\hat{\Pi}_{WW}(0)$. Moreover, $\Delta_G + \hat{\Pi}_{WW}(M^2_W)/M^2_W$ grows only as $\ln(v^2_S)$, so the contribution of the last bracket in (61) vanishes for $v_S \to \infty$. Thus, in this limit one indeed gets (62) and it remains to check that the difference of the Z^0 and $W^\pm$ self-energies approaches the SM form.

For the fermionic contribution to (62) this is clear: for $\hat{\Pi}_{WW}(M^2_W)$ it is exactly as in the SM, and that to $\hat{\Pi}_{Z^0Z^0}(M^2_{Z^0})$ is different, but the difference is only due to Z^0 couplings which, as it follows from (20) and (A.1), approach as $v_S \to \infty$ their SM form. In particular, this means that in the $SU(2)_{\rm L}\times U(1)_Y\times U(1)_E$ model the celebrated contribution $\propto m^2_t/M^2_W$ is present in the $M^2_W \leftrightarrow M^2_{Z^0}$ relation.

Bosonic contributions to $\hat{\Pi}_{WW}(M^2_W)$ and $\hat{\Pi}_{Z^0Z^0}(M^2_{Z^0})$ individually contain terms which grow as $v_S \to \infty$ (the last term in the third line of (D.1) and the $Z'h^0$ contribution to $\hat{\Pi}_{Z^0Z^0}$), but it is easy to check that they cancel out in (62), and the difference $\hat{\Pi}_{WW}(M^2_W)/M^2_W - \hat{\Pi}_{Z^0Z^0}(M^2_{Z^0})/M^2_{Z^0}$ approaches its SM form too.

Thus, we have demonstrated that in the limit $v_S \to \infty$ the finite SM expression for M_{Z^0} is recovered.

6.2 Renormalization scale μ independence of M_{Z^0} at one loop

h^0 tadpoles cancelation

As a first step we show that the h^0 tadpoles $\mathcal{T}_{h^0}$ drop out of the formula (61). The contribution of $\mathcal{T}_{h^0}$ to $2\hat{\Pi}_{Z^0Z^0}$

is

$$2\hat{\Pi}^{h^0\ \mathrm{tad}}_{Z^0Z^0} = 2\left[\frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2} - \frac{\hat{e}}{\hat{s}\hat{c}}e_H g_E\, c's' - \left(\frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2} - e_H^2 g_E^2\right)s'^2\right]\left(-2\hat{v}_H\frac{\mathcal{T}_{h^0}}{M^2_{h^0}}\right). \quad (63)$$

With one-loop accuracy and using the formulae (A.1) this can be rewritten as

$$\left[\frac{M_W^2}{c^2_{(0)}} - \left(\frac{M_W^2}{c^2_{(0)}} - \frac{g_E^2 e_H^2}{\sqrt{2}G_\mathrm{F}}\right)\frac{A_0 - B_0}{\sqrt{\cdots}} + \frac{g_E^2 e_H^2}{\sqrt{2}G_\mathrm{F}} - \frac{e^2_{(0)}}{s^2_{(0)}c^2_{(0)}2G_\mathrm{F}^2}\frac{g_E^2 e_H^2}{\sqrt{\cdots}}\right]\left(-\frac{2}{\hat{v}_H}\frac{\mathcal{T}_{h^0}}{M^2_{h^0}}\right). \quad (64)$$

It is then clear that each term finds in (61) its counterpart with $-\hat{\Pi}^{h^0\ \mathrm{tad}}_{WW}/\hat{M}_W^2 = (2/\hat{v}_H)(\mathcal{T}_{h^0}/M^2_{h^0})$ and exactly the same coefficient.

Contribution proportional to fermion masses squared

Next we consider contributions to $M^2_{Z^0}$ proportional to the fermion masses squared. These are hidden in $\hat{\Pi}_{Z^0Z^0}$, $\hat{\Pi}_{WW}(M_W^2)$ and in $\hat{\Pi}_{WW}(0)$. As usual, they can be isolated by approximating the first two self-energies by $\hat{\Pi}_{Z^0Z^0}(0)$ and $\hat{\Pi}_{WW}(0)$, respectively. From the formula (D.3) we get

$$\Pi^{\mathrm{ferm}}_{Z^0Z^0}(q^2) = -2\sum_f N_c^{(f)}\left(c^{Z^0}_{f\mathrm{L}} - c^{Z^0}_{f\mathrm{R}}\right)^2 m_f^2\, b_0(0,m_f,m_f). \quad (65)$$

Using the couplings (20) and the relations (22) and (A.1) we can write

$$\left(c^{Z^0}_{f\mathrm{L}} - c^{Z^0}_{f\mathrm{R}}\right)^2 = \frac{1}{2}\left\{\frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2} + \left(\frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2} - g_E^2 e_H^2\right)\times\frac{B-A}{\sqrt{\cdots}} + e_H^2 g_E^2 - \frac{\hat{e}^2}{\hat{s}^2\hat{c}^2}\frac{g_E^2 e_H^2 \hat{v}_H^2}{\sqrt{\cdots}}\right\}. \quad (66)$$

This makes it clear that to each term in $2\left[\hat{\Pi}^{\mathrm{ferm}}_{Z^0Z^0}\right]_{\mathrm{mass}}$ there is a corresponding term with $\hat{\Pi}_{WW}$ in the formula (61), so that the divergences proportional to the fermion masses squared properly cancel. Hence, the terms quadratic in the fermion masses arising from "oblique" corrections are finite (and, hence, μ-independent) just as they are in the SM. For the one-loop top–bottom contribution using (50) we get

$$M^2_{Z^0} = \frac{1}{2}\left(A_0 + B_0 - \sqrt{(A_0-B_0)^2 + 4D_0^2}\right) - \left(c^{Z^0}_{f\mathrm{L}} - c^{Z^0}_{f\mathrm{R}}\right)^2\frac{N_c}{16\pi^2}g(m_t,m_b) + \text{other contributions}, \quad (67)$$

and in the limit $v_S \to \infty$ one recovers the SM relation (computed using as input observables M_W, G_F and α_EM).

Remaining fermion contribution and the use of RG equations

The remaining divergent fermionic contribution (D.3) to $\Pi_{Z^0Z^0}$ is proportional to q^2:

$$2\left[\Pi^{\mathrm{ferm}}_{Z^0Z^0}(q^2)\right]^{q^2\ \mathrm{part}}_{\mathrm{div}} = \frac{4}{3}q^2\sum_f N_c^{(f)} \times\left[\left(c^{Z^0}_{f\mathrm{L}}\right)^2 + \left(c^{Z^0}_{f\mathrm{R}}\right)^2\right]\eta_{\mathrm{div}}.$$

Using the couplings (20) and the relations (22) and (A.1) the right hand side takes the form

$$\frac{2}{3}M^2_{Z^0}\left\{\left(1-\frac{A-B}{\sqrt{\cdots}}\right)\times\frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2}\left[2-4\hat{s}^2+8\hat{s}^4+N_c\left(2-4\hat{s}^2+\frac{40}{9}\hat{s}^4\right)\right] + \frac{\hat{e}^2}{\hat{c}^2}\frac{g_E^2 e_H \hat{v}_H^2}{\sqrt{\cdots}}\times\left[2e_l - 2e_{e^c} + N_c\left(-\frac{2}{3}e_q + \frac{4}{3}e_{u^c} - \frac{2}{3}e_{d^c}\right)\right] + \left(1+\frac{A-B}{\sqrt{\cdots}}\right)g_E^2\left[2e_l^2 + e_{e^c}^2 + N_c\left(2e_q^2 + e_{u^c}^2 + e_{d^c}^2\right)\right]\right\}\eta_{\mathrm{div}}. \quad (68)$$

With one-loop accuracy the prefactor of the first line can be transformed into

$$M^2_{Z^0}\left(1-\frac{A-B}{\sqrt{\cdots}}\right) = \frac{M_W^2}{c^2_{(0)}} + \frac{B_0-A_0}{\sqrt{\cdots}}\frac{M_W^2}{c^2_{(0)}} - \frac{1}{2}\frac{e_H^2 g_E^2}{\sqrt{\cdots}}\frac{e^2_{(0)}}{s^2_{(0)}c^2_{(0)}}\frac{1}{2G_\mathrm{F}^2},$$

after which different terms arising from the first line of (68) combine with the appropriate fermionic contributions to

$$-\frac{M_W^2}{c^2_{(0)}}\left[-\frac{\Pi_{WW}(M_W^2)^{q^2\ \mathrm{part}}}{M_W^2} + \frac{s^2_{(0)}}{c^2_{(0)}}\tilde{\Delta}\right]_{\mathrm{div}},$$

in (61) canceling their divergences and the μ dependence exactly as in the SM.

In our renormalization scheme (outlined in Sect. 4) the two other divergent terms in (68) are cut off by the $\overline{\mathrm{MS}}$ procedure. In order to see that $M^2_{Z^0}$ computed at one loop is nevertheless independent of the renormalization scale μ, we have to consider the dependence on μ of $2(M^2_{Z^0})_{(0)}$:

$$(2M^2_{Z^0})_{(0)} = A_0 + B_0(\mu) - \sqrt{\left[A_0 - B_0(\mu)\right]^2 + 4D_0^2(\mu)}. \quad (69)$$

The superscripts 0 on A, B and D mean that the parameters $\hat{e}^2$, $\hat{s}^2$, $\hat{c}^2$ and $\hat{v}_H$ have been expressed in terms of the basic observables $\alpha_{\rm EM}$, M_W and $G_{\rm F}$ to zeroth-order accuracy. The μ dependence is due to the parameters $e_H g_E$, $e_S g_E$ and v_S, which are still the running parameters of the full theory. Using the renormalization group equations (15) and (C.5) for an infinitesimal change of scale μ we have

$$B_0(\mu) = B_0(\mu') + \delta B_1 + \delta B_2 + \delta B_v ,$$

$$4D_0^2(\mu) = 4D_0^2(\mu') + 4\delta D_1^2 + 4\delta D_2^2 ,$$

where

$$\begin{aligned}
\delta B_1 &= \frac{1}{\sqrt{2}G_{\rm F}}\, 2 \\
&\quad \times \left(\frac{2}{3}\sum_f e_H g_E e_f g_E Y_H^Y Y_f^Y + \frac{1}{3} 2 e_H^2 g_E^2 Y_H^Y Y_H^Y\right) \\
&\quad \times g_y^2 \ln\frac{\mu^2}{\mu'^2} , \\
\delta B_2 &= \left(\frac{1}{\sqrt{2}G_{\rm F}} e_H^2 g_E^2 + e_S^2 g_E^2 v_S^2\right) \\
&\quad \times \left(\frac{2}{3}\sum_f e_f^2 g_E^2 + \frac{1}{3} 2 e_H^2 g_E^2 + \frac{1}{3} e_S^2 g_E^2\right) \ln\frac{\mu^2}{\mu'^2} , \\
\delta B_v &= e_S^2 g_E^2 \left(-\frac{3}{2}\lambda_S v_S^2 + 3 e_S^2 g_E^2 v_S^2\right. \\
&\quad \left. -12\frac{g_E^4 e_S^4 v_S^2 + g_E^4 e_S^2 e_H^2 v_H^2}{\lambda_S}\right) \ln\frac{\mu^2}{\mu'^2} , \\
4\delta D_1^2 &= \frac{e_{(0)}^2}{s_{(0)}^2 c_{(0)}^2}\frac{1}{2G_{\rm F}^2} 2 \left(\frac{2}{3}\sum_f e_H g_E e_f g_E Y_H^Y Y_f^Y\right. \\
&\quad \left. + \frac{1}{3} 2 e_H^2 g_E^2 Y_H^Y Y_H^Y\right) g_y^2 \ln\frac{\mu^2}{\mu'^2} , \\
4\delta D_2^2 &= \frac{e_{(0)}^2}{s_{(0)}^2 c_{(0)}^2}\frac{1}{2G_{\rm F}^2} e_H^2 g_E^2 \\
&\quad \times \left(\frac{2}{3}\sum_f e_f^2 g_E^2 + \frac{1}{3} 2 e_H^2 g_E^2 + \frac{1}{3} e_S^2 g_E^2\right) \ln\frac{\mu^2}{\mu'^2} .
\end{aligned} \tag{70}$$

The formula (69) then takes the form

$$\begin{aligned}
(2M_{Z^0}^2)_{(0)} &\approx A_0 + B_0(\mu') - \sqrt{[A_0 - B_0(\mu')]^2 + 4D_0^2(\mu')} \\
&\quad + (\delta B_1 + \delta B_2 + \delta B_v)\left(1 + \frac{A_0 - B_0}{\sqrt{\cdots}}\right) \\
&\quad - \frac{1}{2\sqrt{\cdots}}\left(4\delta D_1^2 + 4\delta D_2^2\right) .
\end{aligned} \tag{71}$$

It is then a matter of some simple algebra to check that the fermion generation number dependent terms in (70) precisely match the $\ln(1/\mu^2)$ proportional terms associated with the two last lines of (68) changing in these terms μ into μ'. Hence, up to one-loop accuracy the entire fermionic contribution to $M_{Z^0}^2$ is renormalization scale independent.

6.2.1 Renormalizations scale independence of the bosonic contribution to $M_{Z^0}^2$

The scale independence of the remaining one-loop contribution can be checked in a similar way (using judiciously the relations collected in Appendix A): part of the divergences with the associated μ dependence explicitly cancels in (61) as a result of expressing $\hat{e}^2$, $\hat{s}^2$, $\hat{c}^2$ and $\hat{v}_H$ in terms of the basic observables $\alpha_{\rm EM}$, M_W and $G_{\rm F}$ with one-loop accuracy. Other divergences are cut off by the $\overline{\rm MS}$ prescription and the explicit renormalization scale dependence is compensated for by the change with μ dictated by the RG of the parameters $e_k g_E$ and v_S in the zeroth-order term $(M_{Z^0}^2)_{(0)}$ (69). Here we only would like to show that the S^0 tadpole contribution to $2\hat{\Pi}_{Z^0Z^0}$ plays a crucial role in the working of the scheme [17].

The couplings of S^0 to S^0S^0 and to $G'G'$, G^0G^0 can easily be computed.[8] For the S^0 tadpole we then get

$$\begin{aligned}
\mathcal{T}_{S^0} &= \frac{3}{4}\lambda_S v_S a(M_{S^0}) + \frac{1}{4}\lambda_S v_S \tilde{c}'^2 a(M_{G'}) \\
&\quad + \frac{1}{4}\lambda_S v_S \tilde{s}'^2 a(M_{G^0}) + 3 g_E^2 e_S^2 v_S \\
&\quad \times \left[c'^2 M_{Z'}^2\left(\ln\frac{M_{Z'}^2}{\mu^2} - \frac{1}{3}\right) + s'^2 M_{Z^0}^2\left(\ln\frac{M_{Z^0}^2}{\mu^2} - \frac{1}{3}\right)\right] ,
\end{aligned}$$

where $\tilde{c}'$ and $\tilde{s}'$ are the mixing angles of G^0 and G'. $\tilde{c}'$ and $\tilde{s}'$ are different from c' and s', but still one has the usual relations $M_{G^0}^2 = \xi M_{Z^0}^2$ and $M_{G'}^2 = \xi M_{Z'}^2$. The S^0 mass is $M_{S^0}^2 = \frac{1}{2}\lambda_S v_S^2$. As usually we work in the Feynman gauge, $\xi = 1$.

The S^0 tadpole gives

$$\begin{aligned}
2\hat{\Pi}_{Z^0Z^0}^{S^0\ \rm tad} &= 2\cdot 2 g_E^2 e_S^2 v_S\ s'^2\left(-\frac{\mathcal{T}_{S^0}}{M_{S^0}^2}\right) \\
&= -4 g_E^2 e_S^2 v_S\left(1 + \frac{A - B}{\sqrt{\cdots}}\right)\frac{1}{\lambda_S v_S^2} \\
&\quad \times \left\{\frac{3}{4}\lambda_S v_S \frac{1}{2}\lambda_S v_S^2 + \frac{1}{4}\lambda_S v_S g_E^2 e_S^2 v_S^2\right. \\
&\quad \left. + 3 g_E^2 e_S^2 v_S\left(g_E^2 e_S^2 v_S^2 + g_E^2 e_H^2 v_H^2\right)\right\} \ln\frac{1}{\mu^2} + \ldots ,
\end{aligned}$$

where we have used the relations $s'^2 M_{Z^0}^2 + c'^2 M_{Z'}^2 = g_E^2 e_S^2 v_S^2 + g_E^2 e_H^2 v_H^2$ and $\tilde{s}'^2 M_{Z^0}^2 + \tilde{c}'^2 M_{Z'}^2 = e_S^2 g_E^2 v_S^2$.

From $(2M_{Z^0}^2)^{\rm tree}$, see (69), we have instead

$$(2M_{Z^0}^2)^{\rm tree} \supset \left(1 + \frac{A_0 - B_0}{\sqrt{\cdots}}\right)\delta B_v .$$

---

[8] As explained in Appendix C, in order to simplify the formulae we assume that at the scale we are working the scalar potential is the sum $V = V_H(H) + V_S(S)$. The physical Higgs scalars S^0 and h^0 are then pure real parts of the singlet S and of the neutral component of the doublet H. The S^0 does not couple then to h^0h^0.

This explicitly shows that in the S^0 tadpole contribution the scale μ is properly replaced by μ' in the terms $\propto \lambda_S$ and $\propto (1/\lambda_S)$ (as we have checked, the λ_S independent terms in $\mathcal{T}_{S^0}$ combine with the bosonic contribution $\hat{\Pi}_{Z^0Z^0}$).

We have shown that in the one-loop expression for $M^2_{Z^0}$, consistent with the Appelquist–Carrazzone decoupling, the explicit renormalization scale dependence is only in terms suppressed by inverse powers of v_S. Moreover, the whole expression is in fact renormalization scale independent, if one takes into account the μ dependence of the RG running of the parameters in the tree-level term.

7 On-shell Z^0 couplings to fermions

In this section we briefly consider the parameter ρ defined in terms of the physical Z^0 and $W^\pm$ masses and the Weinberg angle:

$$\rho = \frac{M^2_W}{M^2_{Z^0}\left(1-\sin^2\theta^\ell_{\rm eff}\right)}\,, \tag{72}$$

where $\sin^2\theta^\ell_{\rm eff}$ is defined by the form (24) of the effective coupling of the on-shell Z^0 to the fermions (we take leptons for definiteness):

$$\mathcal{L}^{Z^0 f\bar{f}\ \text{on-shell}}_{\rm eff} = \bar{\psi}_l\gamma^\mu(F_{\rm L}\mathbf{P}_{\rm L}+F_{\rm R}\mathbf{P}_{\rm R})\psi_l Z^0_\mu\,. \tag{73}$$

Comparison of (73) with (24) gives $\sin^2\theta^\ell_{\rm eff} = F_{\rm R}/2(F_{\rm R}-F_{\rm L})$. For the form factors $F_{\rm L,R}$ we have the formulae

$$\begin{aligned}F_{\rm L,R} = &-c^{Z^0}_{\ell{\rm L,R}} - \frac{1}{2}\hat{\Pi}'_{Z^0Z^0}\left(M^2_{Z^0}\right)c^{Z^0}_{\ell{\rm L,R}}\\ &+\hat{e}\frac{\hat{\Pi}_{Z^0\gamma}\left(M^2_{Z^0}\right)}{M^2_{Z^0}} - \frac{\hat{\Pi}_{Z^0Z'}\left(M^2_{Z^0}\right)}{M^2_{Z^0}-M^2_{Z'}}c^{Z'}_{\ell{\rm L,R}}+\dots\end{aligned} \tag{74}$$

Since we are interested only in the dominant universal top–bottom contribution, we have not written down the genuine vertex corrections, nor the final fermion self energies.

Expressing the running coupling constants in $c^{Z^0}_{\ell{\rm L,R}}$ in terms of M^2_W, $G_{\rm F}$ and $\alpha_{\rm EM}$ with one-loop accuracy we find

$$\begin{aligned}c^{Z^0}_{\ell{\rm R}} = &\,e_{(0)}\frac{s_{(0)}}{c_{(0)}}\left\{1-\frac{1}{2}\hat{\hat{\Pi}}_\gamma(0) - \frac{\alpha_{\rm EM}}{2\pi}\ln\frac{M^2_W}{\mu^2}+\frac{1}{2c^2_{(0)}}\varDelta\right\}c'_{(0)}\\ &- e_{\ell^c}g_E s'_{(0)} + e_{(0)}\frac{s_{(0)}}{c_{(0)}}\,\delta c' - e_{\ell^c}g_E\,\delta s'\,,\\ c^{Z^0}_{\ell{\rm L}} = &-\frac{e_{(0)}}{2s_{(0)}c_{(0)}}\left\{1-\frac{1}{2}\hat{\hat{\Pi}}_\gamma(0)-\frac{\alpha_{\rm EM}}{2\pi}\right.\\ &\left.\times\ln\frac{M^2_W}{\mu^2}+\frac{s^2_{(0)}-c^2_{(0)}}{2c^2_{(0)}}\varDelta\right\}c'_{(0)}\\ &+e_{(0)}\frac{s_{(0)}}{c_{(0)}}\left\{1-\frac{1}{2}\hat{\hat{\Pi}}_\gamma(0)-\frac{\alpha_{\rm EM}}{2\pi}\right.\\ &\left.\times\ln\frac{M^2_W}{\mu^2}+\frac{1}{2c^2_{(0)}}\varDelta\right\}c'_{(0)}\\ &+e_\ell g_E s'_{(0)} - \frac{e_{(0)}}{2s_{(0)}c_{(0)}}\left(1-2s^2_{(0)}\right)\delta c' + e_\ell g_E\,\delta s'\,,\end{aligned} \tag{75}$$

where $\varDelta$ is given in (47). We have also introduced $\delta c'$ and $\delta s'$ because the original c' and s' depend on $\hat{e}$, $\hat{s}$, $\hat{c}$ and $\hat{v}^2_H$. The quantities $c'_{(0)}$ and $s'_{(0)}$ are then given by the same expressions as c' and s' but with $\hat{e}$, $\hat{s}$, $\hat{c}$ and $\hat{v}^2_H$ replaced by $e_{(0)}$, $s_{(0)}$, $c_{(0)}$ and $1/\sqrt{2}G_{\rm F}$, respectively.

In our renormalization scheme the form factors $F_{\rm L,R}$ given by (74) and (75) are finite if the $\overline{\rm MS}$ scheme is employed. Moreover their parts non-vanishing as $v_S\to\infty$ are renormalization scale independent (i.e. they are just finite) and the explicit μ dependence of the one-loop terms is compensated for by the change of the running parameters e_Hg_E, $e_\ell g_E$, $e_{\ell^c}g_E$ and v_S entering the zeroth-order contributions.

For $\delta c'$ and $\delta s'$ we find

$$\begin{aligned}\delta s' = -\frac{c'_{(0)}}{4s'_{(0)}}\delta c' = &\,\frac{1}{4s'_{(0)}}\frac{1}{(\sqrt{\dots})^3}\\ &\times\left[4D_0^2(\delta A-\delta B)-(A_0-B_0)4D_0\delta D\right]\\ =&\,\frac{c'_{(0)}}{(\sqrt{\dots})^2}\frac{e_{(0)}}{2s_{(0)}c_{(0)}}\frac{e_He^2_Sg^3_Ev^2_S}{\sqrt{2}G_{\rm F}}\\ &\times\left[-\frac{\hat{\Pi}_{WW}(0)}{M^2_W}\right]+\dots\,,\end{aligned} \tag{76}$$

where in the second line, in order to isolate the dominant top–bottom contributions to the form factors $F_{\rm L}$ and $F_{\rm R}$, we have isolated only the term with $\hat{\Pi}_{WW}(0)$. Combining this with

$$\begin{aligned}\hat{\Pi}_{Z^0Z'}\left(M^2_{Z^0}\right)\approx&\,\hat{\Pi}_{Z^0Z'}(0)\\ =&-\sum_f(c^{Z^0}_{f{\rm L}}-c^{Z^0}_{f{\rm R}})(c^{Z'}_{f{\rm L}}-c^{Z'}_{f{\rm R}})2N^{(f)}_c m^2_f\\ &\times b_0(0,m_f,m_f)\\ =&-\frac{1}{(\sqrt{\dots})}\frac{\hat{e}}{2\hat{s}\hat{c}}e_He^2_Sg^3_Ev^2_S\\ &\times\sum_f 2N^{(f)}_c m^2_f b_0(0,m_f,m_f)\end{aligned} \tag{77}$$

(where again we have used the results (20), (21) and (A.2)) and using the fact that $M^2_{Z^0}-M^2_{Z'} = -\sqrt{\dots}$ we find

$$F^{t,b}_{\rm L,R}\approx-\frac{1}{(\sqrt{\dots})^2}\frac{\hat{e}}{2\hat{s}\hat{c}}e_He^2_Sg^3_Ev^2_Sc^{Z'}_{\ell L,R}\frac{N_c}{16\pi^2}g(m_t,m_b)\,. \tag{78}$$

Since $(\sqrt{\dots})^2\equiv(A_0-B_0)^2+4D_0^2\sim v^4_S$ as $v_S\to\infty$, this contribution is explicitly suppressed in this limit. It is easy to see that the expressions for $F_{\rm L}$ and $F_{\rm R}$, see (74) and (75), do not involve any other contributions proportional to m^2_t and m^2_b and, therefore, no contributions $\propto m^2_t/M^2_W$ enter

$\sin^2\theta^\ell_{\rm eff}$ at one loop.[9] Since we have already shown that for $v_S\to\infty$ one recovers also the SM expression for M_{Z^0}, and we conclude that in the $U(1)_Y\times U(1)_E$ model

$$\rho=\frac{M_W^2}{M_{Z^0}^2\left(1-\sin^2\theta^\ell_{\rm eff}\right)}=1+\frac{N_c}{16\pi^2}\sqrt{2}G_{\rm F}g(m_t,m_b)+\mathcal{O}\left(m_t^2/v_S^2\right)+\dots\,, \qquad (79)$$

where the dots stand for other SM contributions as well as for other terms suppressed in the limit $v_S\to\infty$ (also those arising from the tree-level contribution contributon to ρ, see (33)). Similar result can be proven also for ρ_{Zf} defined by the effective Lagrangian (24).

It should be stressed that unlike $\rho_{\rm low}$ to which one-loop corrections have been computed in Sect. 5, the parameter ρ defined in (72) is not equal to unity at the tree level. Therefore the one-loop result for ρ does depend on the renormalization scheme and in particular on the chosen set of input observables. This observation is helpful in understanding the apparent discrepancy of our results with the claim of [5–7] that in models like the one considered here the contribution to ρ proportonal to m_t^2/M_W^2 is absent. References [5–7] use $\sin^2\theta^\ell_{\rm eff}$ as one of the input observables and then, as we have commented earlier, the explicit Appelquist–Carrazzone decoupling is lost. However, our point is that the renormalization scheme can be chosen in such a way that new physics effects can be treated as corrections to the well established SM resuls.

8 Conclusions

In this paper we have discussed some technical aspects related to the $U(1)_E$ extension of the standard electroweak theory. We have elucidated the correct treatment of the additional coupling constants and presented the one-loop renormalization group equations in a form adapted to practical calculations. Furthermore we have proposed a renormalization scheme employing as in the SM only three input observables (for technical convenience we have chosen to work with M_W, $G_{\rm F}$ and $\alpha_{\rm EM}$ instead of the customary set M_{Z^0}, $G_{\rm F}$ and $\alpha_{\rm EM}$) which has the virtue of making the decoupling of heavy Z' effects manifest. To demonstrate this we have computed the parameter ρ defined either in terms of the low energy neutrino scattering processes or in terms of physical M_W^2, $M_{Z^0}^2$ and $\sin^2\theta^l_{\rm eff}$ as measured in $Z^0\to l^+l^-$. In addition, in both cases we have shown explicitly in a renormalization scheme in which the Appelquist–Carrazzone decoupling is manifest that the $\propto G_{\rm F}m_t^2$ contribution to the ρ parameters is present and up to terms vanishing as $M_{Z'}\to\infty$ takes the form as in the SM. Our calculation supports therefore similar an observation made in [9] a long time ago.

Our choice of M_W, $G_{\rm F}$ and $\alpha_{\rm EM}$ as input observables instead of the commonly used set M_Z, $G_{\rm F}$ and $\alpha_{\rm EM}$ was dictated by the desire of demonstrating crucial aspects of our renormalization scheme (in particular the role of the renormalization group equations in proving scale independence of the computed observables) analytically. We have checked, however, that the explicit decoupling of heavy Z' effects (that the expressions for the electroweak observables approach their SM form for $v_S\propto M_{Z'}\to\infty$), do not depend on whether one uses M_W or M_Z.

The Appelquist–Carrazzone decoupling offers a possibility of a systematic inclusion of all large logarithmic $\sim[\ln(M_{Z'}/M_{Z^0})]^n$ corrections by taking into account the RG running of the Wilson coeffcients of non-renormalizable operators generated by decoupling of the heavy Z' sector.

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Appendix A: Useful formulae

The mass matrix of Z^0 and Z' which arises as a 2×2 submatrix after rotating (16) by the angle $\theta_{\rm W}$ reads

$$\begin{pmatrix}\frac{1}{4}\left(g_y^2+g_2^2\right)v_H^2 & -\frac{1}{2}\sqrt{g_y^2+g_2^2}\,g_E e_H v_H^2\\ -\frac{1}{2}\sqrt{g_y^2+g_2^2}\,g_E e_H v_H^2 & e_H^2 g_E^2 v_H^2+e_S^2 g_E^2 v_S^2\end{pmatrix}=\begin{pmatrix}A & D\\ D & B\end{pmatrix}.$$

It is diagonalized by the rotation by the angle θ' determined from (18). For s'^2, c'^2 and $s'c'$ one derives the following useful expressions:

$$\begin{aligned}s'^2&=\frac{1}{2}\left(1+\frac{A-B}{\sqrt{(A-B)^2+4D^2}}\right)=\frac{1}{4}\frac{g_y^2+g_2^2}{g_E^2}\frac{e_H^2v_H^4}{e_S^4v_S^4}+\dots\,,\\ c'^2&=\frac{1}{2}\left(1-\frac{A-B}{\sqrt{(A-B)^2+4D^2}}\right)=1-\frac{1}{8}\frac{g_y^2+g_2^2}{g_E^2}\frac{e_H^2v_H^4}{e_S^4v_S^4}+\dots\,,\\ s'c'&=\frac{-D}{\sqrt{(A-B)^2+4D^2}}=\frac{1}{2}\frac{\sqrt{g_y^2+g_2^2}}{g_E}\frac{e_Hv_H^2}{e_S^2v_S^2}+\dots\end{aligned} \qquad (A.1)$$

Other useful expressions are

$$\begin{aligned}\frac{s'^2}{M_{Z^0}^2}+\frac{c'^2}{M_{Z'}^2}&=\frac{1}{M_{Z^0}^2M_{Z'}^2}\left(\frac{e^2}{4s^2c^2}v_H^2\right)\\ \frac{c'^2}{M_{Z^0}^2}+\frac{s'^2}{M_{Z'}^2}&=\frac{1}{M_{Z^0}^2M_{Z'}^2}\left(g_E^2e_H^2v_H^2+g_E^2e_S^2v_S^2\right)\\ s'c'\left(\frac{1}{M_{Z^0}^2}-\frac{1}{M_{Z'}^2}\right)&=\frac{1}{M_{Z^0}^2M_{Z'}^2}\left(\frac{e}{2sc}g_Ee_Hv_H^2\right)\end{aligned} \qquad (A.2)$$

[9] In the SM the form factors $F_{\rm L}$ and $F_{\rm R}$ do not receive any such contribution if the scheme based on M_W, $G_{\rm F}$ and $\alpha_{\rm EM}$ as input observables is employed.

and

$$M_{Z^0}^2 M_{Z'}^2 = \frac{e^2}{4s^2c^2} g_E^2 e_S^2 v_H^2 v_S^2 \,. \tag{A.3}$$

Still other useful relations are

$$\begin{aligned} c'^2 M_{Z^0}^2 &= Ac'^2 + Ds'c' \,, \\ s'^2 M_{Z^0}^2 &= Bs'^2 + Ds'c' \,, \\ c'^2 M_{Z'}^2 &= Bc'^2 - Ds'c' \,, \\ s'^2 M_{Z'}^2 &= As'^2 - Ds'c' \,. \end{aligned} \tag{A.4}$$

Appendix B: Calculation of the input observables $\alpha_{\rm EM}$, $G_{\rm F}$ and M_W

Here we outline the calculation in the $SU(2)_{\rm L} \times U(1)_Y \times U(1)_E$ model of the basic input observables $\alpha_{\rm EM}$, $G_{\rm F}$ and M_W. The formula for M_W is simple:

$$M_W^2 = \frac{\hat{e}^2}{4\hat{s}^2} \hat{v}_H^2 + \hat{\Pi}_{WW}\left(M_W^2\right) \,, \tag{B.1}$$

where $\hat{\Pi}_{WW}(M_W^2)$ includes in principle also the tadpole contribution. Expressions for $\alpha_{\rm EM}$ and $G_{\rm F}$ are derived below.

B.1 Calculation of $\delta\alpha_{\rm EM}$

This is most easily computed using the effective Lagrangian technique [16]. Below the electroweak scale (the renormalizable part of) the effective Lagrangian for electromagnetic interactions has the form

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}(1+\delta z_\gamma) f_{\mu\nu} f^{\mu\nu} \\ &+ \left(1+\delta z_2^{\rm L}\right) \bar{\psi}_e {\rm i}\, \partial\!\!\!/ \, \mathbf{P}_{\rm L} \psi_e \\ &- \left(e + \delta e + \hat{e}\, \delta z_2^{\rm L} + \frac{1}{2}\hat{e}\delta z_\gamma\right) \bar{\psi}_e\, q_e\, A\!\!\!/\, \mathbf{P}_{\rm L}\psi_e \\ &+ \left(1+\delta z_2^{\rm R}\right) \bar{\psi}_e {\rm i}\, \partial\!\!\!/ \, \mathbf{P}_{\rm R} \psi_e \\ &- \left(\hat{e} + \delta e + \hat{e}\, \delta z_2^{\rm R} + \frac{1}{2}\hat{e}\, \delta z_\gamma\right) \bar{\psi}_e\, q_e\, A\!\!\!/\, \mathbf{P}_{\rm R}\psi_e \\ &+ \text{counterterms}\,. \end{aligned} \tag{B.2}$$

$\hat{e} + \delta e$ is the electromagnetic coupling of QED at the scale just below the Fermi scale threshold; it can easily be related to $\alpha_{\rm EM}$ via the RG running.

The factors $\delta z_2^{\rm L}$ and $\delta z_2^{\rm R}$ are such that they reproduce at the tree level contributions of virtual W, Z^0 and Z' to the electron self-energies (computed at zero momentum). Similarly,

$$\delta z_\gamma = -[\tilde{\Pi}_\gamma(0)]_{W,G^+,f} \tag{B.3}$$

reproduces at the tree level the vacuum polarization due to decoupled heavy particles $W^\pm$ and top quark.

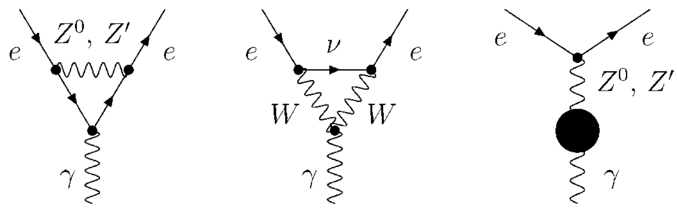

Fig. 2. Corrections to the photon–electron vertex in a model with extra $U(1)$. The external line momenta can be off-shell but must be $\ll M_Z$

The vertex corrections determining the combinations $\delta e + \hat{e}\, \delta z_2^{\rm L,R} + \frac{1}{2}\hat{e}\, \delta z_\gamma$ are shown in Fig. 2. Owing to the $U(1)_Y$ and $U(1)_E$ Ward identities the Z^0 and Z' contributions to δe are exactly canceled by the Z^0 and Z' contributions to $\delta z_2^{\rm L}$ and $\delta z_2^{\rm R}$, respectively. The second diagram in Fig. 2 is exactly as in the SM and combines with the W contribution to $\delta z_2^{\rm L}$. As a result from the photon coupling to left-chiral electrons one gets

$$\begin{aligned} \delta e = &\frac{1}{2}\hat{e}\, \tilde{\Pi}_\gamma(0) + c_{e{\rm L}}^{Z^0} \frac{\hat{\Pi}_{\gamma Z^0}(0)}{M_{Z^0}^2} + c_{e{\rm L}}^{Z'} \frac{\hat{\Pi}_{\gamma Z'}(0)}{M_{Z'}^2} \\ &- \frac{\hat{e}^3}{16\pi^2 \hat{s}^2} \left(\eta_{\rm div} + \ln\frac{\hat{M}_W^2}{\mu^2}\right) . \end{aligned}$$

The self-energies $\hat{\Pi}_{\gamma Z^0}(0)$ and $\hat{\Pi}_{\gamma Z'}(0)$ receive contributions only from the virtual W^+W^- and $W^\pm G^\mp$ pairs. We get

$$\begin{aligned} \delta e = &\frac{1}{2}\hat{e}\, \tilde{\Pi}_\gamma(0) - \frac{1}{16\pi^2} \frac{\hat{e}^3}{\hat{s}^2} \left(\eta_{\rm div} + \ln\frac{\hat{M}_W^2}{\mu^2}\right) \\ &+ \frac{1}{16\pi^2} 2c_{e{\rm L}}^{Z^0} \left[-\hat{e}^2 \frac{\hat{c}}{\hat{s}} c' - \hat{e}\left(\hat{e}\frac{\hat{s}}{\hat{c}} c' - 2e_H g_E s'\right)\right] \\ &\times \frac{\hat{M}_W^2}{M_{Z^0}^2} \ln\frac{\hat{M}_W^2}{\mu^2} + \frac{1}{16\pi^2} 2c_{e{\rm L}}^{Z'} \\ &\times \left[\hat{e}^2 \frac{\hat{c}}{\hat{s}} s' + \hat{e}\left(\hat{e}\frac{\hat{s}}{\hat{c}} s' + 2e_H g_E c'\right)\right] \frac{\hat{M}_W^2}{M_{Z'}^2} \ln\frac{\hat{M}_W^2}{\mu^2} \,. \end{aligned}$$

By using (A.2) and (A.3) this can be reduced to

$$\delta e = \frac{1}{2}\hat{e}\, \tilde{\Pi}_\gamma(0) - \frac{\hat{e}^3}{8\pi^2} \left(\eta_{\rm div} + \ln\frac{\hat{M}_W^2}{\mu^2}\right) ,$$

which (as could be expected) is the same as in the SM. The same result is obtained by considering the photon coupling to a right-chiral electron.

B.2 Calculation of $\delta G_{\rm F}$

Calculation of $\delta G_{\rm F}$ proceeds as in the SM. The only modification is that there are additional box diagrams with Z' and in addition the W boson self-energy $\Pi_{WW}(q^2)$ as well as the self-energies of external line fermions are modified by the presence of Z' (there are contributions from the virtual

Z' and the couplings of Z^0 are modified). Still the formula takes the form

$$G_{\rm F} = \frac{1}{\sqrt{2}\hat{v}_H^2}(1+\Delta_G) = \frac{\hat{e}^2}{4\sqrt{2}\hat{s}^2\hat{M}_W^2}(1+\Delta_G)\,,$$

with Δ_G given by (B.4):

$$\Delta_G = -\frac{\hat{\Pi}_{WW}(0)}{\hat{M}_W^2} + B_{W\gamma} + B_{WZ^0} + B_{WZ'} + 2\hat{\Lambda} + \hat{\Sigma}_{e\rm L} + \hat{\Sigma}_{\nu\rm L}\,. \tag{B.4}$$

Here $B_{W\gamma}$ is the contribution (in units of the tree level W exchange) of the $W\gamma$ box with a subtracted contribution of the photonic vertex correction to the tree-level diagram in the low energy effective four-Fermi theory of μ^- decay:

$$B_{W\gamma} = \frac{\hat{e}^2}{16\pi^2}\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{M_W^2}{\mu^2}\right)$$

(this contribution is the same as in the SM), and B_{WZ^0} and $B_{WZ'}$ denote the contributions of the box diagrams with WZ^0 and WZ', respectively:

$$B_{WZ^0} = \frac{1}{16\pi^2}\left[\left(c_{e\rm L}^{Z^0}\right)^2 + \left(c_{\nu\rm L}^{Z^0}\right)^2 - 8\,c_{e\rm L}^{Z^0}c_{\nu L}^{Z^0}\right] \times \frac{M_W^2}{M_W^2 - M_{Z^0}^2}\ln\frac{M_W^2}{M_{Z^0}^2}\,,$$

and $B_{WZ'}$ is given by a similar expression with $c_{e,\nu L}^{Z^0} \to c_{e,\nu L}^{Z'}$ and $M_{Z^0}^2 \to M_{Z'}^2$.

For the contributions $\hat{\Lambda}^{(i)}$ of individual diagrams to the vertex corrections $\hat{\Lambda} = (1/16\pi^2)\sum_i \hat{\Lambda}^{(i)}$ one finds

$$\hat{\Lambda}^{Z^0 e\nu} = -c_{e\rm L}^{Z^0}c_{\nu\rm L}^{Z^0}\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_{Z^0}^2}{\mu^2}\right)\,,$$

$$\hat{\Lambda}^{Z' e\nu} = -c_{e\rm L}^{Z'}c_{\nu\rm L}^{Z'}\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_{Z'}^2}{\mu^2}\right)\,,$$

$$\hat{\Lambda}^{\nu WZ^0} = -3c_{\nu\rm L}^{Z^0}\left(\hat{e}\frac{\hat{c}}{\hat{s}}c'\right) \times\left(\eta_{\rm div} - \frac{5}{6} + \ln\frac{\hat{M}_W^2}{\mu^2} + \frac{\hat{M}_{Z^0}^2}{\hat{M}_{Z^0}^2 - \hat{M}_W^2}\ln\frac{\hat{M}_{Z^0}^2}{\hat{M}_W^2}\right)\,,$$

$$\hat{\Lambda}^{\nu WZ'} = -3c_{\nu\rm L}^{Z'}\left(-\hat{e}\frac{\hat{c}}{\hat{s}}s'\right) \times\left(\eta_{\rm div} - \frac{5}{6} + \ln\frac{\hat{M}_W^2}{\mu^2} + \frac{\hat{M}_{Z'}^2}{\hat{M}_{Z'}^2 - \hat{M}_W^2}\ln\frac{\hat{M}_{Z'}^2}{\hat{M}_W^2}\right)\,,$$

$$\hat{\Lambda}^{eZ^0W} = 3c_{e\rm L}^{Z^0}\left(\hat{e}\frac{\hat{c}}{\hat{s}}c'\right) \times\left(\eta_{\rm div} - \frac{5}{6} + \ln\frac{\hat{M}_W^2}{\mu^2} + \frac{\hat{M}_{Z^0}^2}{\hat{M}_{Z^0}^2 - \hat{M}_W^2}\ln\frac{\hat{M}_{Z^0}^2}{\hat{M}_W^2}\right)\,,$$

$$\hat{\Lambda}^{eZ'W} = 3c_{e\rm L}^{Z'}\left(-\hat{e}\frac{\hat{c}}{\hat{s}}s'\right) \times\left(\eta_{\rm div} - \frac{5}{6} + \ln\frac{\hat{M}_W^2}{\mu^2} + \frac{\hat{M}_{Z'}^2}{\hat{M}_{Z'}^2 - \hat{M}_W^2}\ln\frac{\hat{M}_{Z'}^2}{\hat{M}_W^2}\right)\,,$$

$$\hat{\Lambda}^{e\gamma W} = -3\hat{e}^2\left(\eta_{\rm div} - \frac{5}{6} + \ln\frac{\hat{M}_W^2}{\mu^2}\right)\,,$$

so that the divergent part of $\hat{\Lambda}$ is

$$\hat{\Lambda}_{\rm div} = \frac{1}{16\pi^2}\left(\hat{e}^2\frac{1-2\hat{s}^2-12c^2}{4\hat{s}^2\hat{c}^2} - e_l^2 g_E^2\right)\eta_{\rm div}\,.$$

Finally, for the self-energies $\hat{\Sigma}_{\nu\rm L}$ and $\hat{\Sigma}_{e\rm L}$ of the left-chiral electron and neutrino, respectively, one gets

$$16\pi^2\hat{\Sigma}_{\nu\rm L} = \frac{\hat{e}^2}{2\hat{s}^2}\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_W^2}{\mu^2}\right) + \left(c_{\nu L}^{Z^0}\right)^2\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_{Z^0}^2}{\mu^2}\right) + \left(c_{\nu\rm L}^{Z'}\right)^2\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_{Z'}^2}{\mu^2}\right)\,,$$

$$16\pi^2\hat{\Sigma}_{e\rm L} = \frac{\hat{e}^2}{2\hat{s}^2}\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_W^2}{\mu^2}\right) + \left(c_{e\rm L}^{Z^0}\right)^2\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_{Z^0}^2}{\mu^2}\right) + \left(c_{e\rm L}^{Z'}\right)^2\left(\eta_{\rm div} + \frac{1}{2} + \ln\frac{\hat{M}_{Z'}^2}{\mu^2}\right)\,,$$

with the divergent part

$$\left(\hat{\Sigma}_{\nu\rm L} + \hat{\Sigma}_{eL}\right)_{\rm div} = \frac{1}{16\pi^2} \times\left(\frac{\hat{e}^2}{\hat{s}^2} + \frac{\hat{e}^2}{4\hat{s}^2\hat{c}^2}[1+(1-2\hat{s}^2)^2] + 2e_l^2 g_E^2\right)\eta_{\rm div}\,.$$

Collecting all divergent parts, we get for the box, vertex and self-energy corrections exactly the same divergent part as in the SM

$$\left(B_{\rm boxes} + 2\hat{\Lambda} + \hat{\Sigma}_{e\rm L} + \hat{\Sigma}_{\nu\rm L}\right)_{\rm div} = -\frac{\hat{e}^2}{16\pi^2}\frac{4}{\hat{s}^2}\eta_{\rm div}\,. \tag{B.5}$$

Appendix C: RG equation for v_S

The most general scalar field potential in the model considered in this paper is

$$V = m_S^2 S^\star S + \frac{\lambda_S}{4}(S^\star S)^2 + m_H^2 H^\dagger H + \frac{\lambda_H}{4}(H^\dagger H)^2 + \kappa(S^\star S)(H^\dagger H)\,.$$

In order to simplify the formulae we have assumed that at one particular renormalization scale μ, at which we chose to work, $\kappa(\mu)=0$. However, to derive the renormalization group equation for v_S one has to keep κ. With

$$S=\frac{1}{\sqrt{2}}(v_S+S^0+\mathrm{i}G_S)\,,\quad H=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}G^+\\ v_H+h^0+\mathrm{i}G_H\end{pmatrix} \tag{C.1}$$

(where h^0 and S^0 are the physical Higgs scalars and G_H and G_S are the fields whose appropriate linear combinations G^0 and G' become the longitudinal components of the massive Z^0 and Z'), the formulae determining v_S^2 and v_H^2 read

$$\begin{aligned} m_H^2+\frac{1}{4}\lambda_H v_H^2+\frac{1}{2}\kappa v_S^2=0\,,\\ m_S^2+\frac{1}{4}\lambda_S v_S^2+\frac{1}{2}\kappa v_H^2=0\,. \end{aligned} \tag{C.2}$$

Differentiating the second one with respect to μ we get at $\kappa=0$

$$\mu\frac{\mathrm{d}v_S^2}{\mathrm{d}\mu}=-\frac{4}{\lambda_S}\left(\mu\frac{\mathrm{d}m_S^2}{\mathrm{d}\mu}+\frac{1}{4}v_S^2\,\mu\frac{\mathrm{d}\lambda_S}{\mathrm{d}\mu}+\frac{1}{2}v_H^2\,\mu\frac{\mathrm{d}\kappa}{\mathrm{d}\mu}\right)\,. \tag{C.3}$$

Thus, to find the derivative of v_S^2 at the scale μ, such that $\kappa(\mu)=0$ we need to get also $\mathrm{d}\kappa/\mathrm{d}t$. Calculating derivatives appearing in the right hand side of (C.3) is standard:

$$\begin{aligned} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda_S&=2\epsilon\lambda_S+5\lambda_S^2-12\lambda_S g_E^2e_S^2+24g_E^4e_S^4\,,\\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}m_S^2&=m_S^2\left(2\lambda_S-6g_E^2e_S^2\right)\,,\\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\kappa&=12g_E^4e_S^2e_H^2\,. \end{aligned} \tag{C.4}$$

Using these results and (C.3) it is easy to derive

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}v_S^2=v_S^2\left(-3\lambda_S+6g_E^2e_S^2\right)-24\frac{g_E^4e_S^4v_S^2+g_E^4e_S^2e_H^2v_H^2}{\lambda_S}\,. \tag{C.5}$$

Appendix D: Vector boson self-energies

The fermionic one-loop contribution to $\Pi_{WW}(q^2)$ in $SU(2)\times U(1)_E\times U(1)_Y$ is as in the SM. For the bosonic part of $\Pi_{WW}(q^2)$ we have

$$\begin{aligned} &-\frac{\hat e^2}{\hat s^2}\tilde A\left(q^2,\hat M_W,\hat M_{h^0}\right)-\frac{\hat e^2}{\hat s^2}\tilde A\left(q^2,\hat M_W,\hat M_{Z^0}\right)\\ &+\frac{\hat e^2}{\hat s^2}\hat M_W^2 b_0\left(q^2,\hat M_W,\hat M_{h^0}\right)+\hat e^2\hat M_W^2 b_0\left(q^2,\hat M_W,0\right)\\ &+\left(-\hat e\frac{\hat s}{\hat c}c'+2e_Hg_Es'\right)^2\hat M_W^2 b_0\left(q^2,\hat M_W,\hat M_{Z^0}\right)\\ &+\left(\hat e\frac{\hat s}{\hat c}s'+2e_Hg_Ec'\right)^2\hat M_W^2 b_0\left(q^2,\hat M_W,\hat M_{Z'}\right)\\ &-\hat e^2\frac{\hat c^2}{\hat s^2}{c'}^2\left[8\tilde A\left(q^2,\hat M_W,\hat M_{Z^0}\right)+\left(4q^2+\hat M_W^2+\hat M_{Z^0}^2\right)\right.\\ &\left.\times b_0\left(q^2,\hat M_W,\hat M_{Z^0}\right)-\frac{2}{3}\frac{q^2}{16\pi^2}\right]\\ &-\hat e^2\frac{\hat c^2}{\hat s^2}{s'}^2\left[8\tilde A\left(q^2,\hat M_W,\hat M_{Z'}\right)+\left(4q^2+\hat M_W^2+\hat M_{Z'}^2\right)\right.\\ &\left.\times b_0\left(q^2,\hat M_W,\hat M_{Z'}\right)-\frac{2}{3}\frac{q^2}{16\pi^2}\right]\\ &-\hat e^2\left[8\tilde A\left(q^2,\hat M_W,0\right)+\left(4q^2+\hat M_W^2\right)b_0\left(q^2,\hat M_W,0\right)\right.\\ &\left.-\frac{2}{3}q^2\frac{q^2}{16\pi^2}\right]\,. \end{aligned} \tag{D.1}$$

The divergent part of this contribution taken at $q^2=0$ is

$$16\pi^2[\Pi_{WW}(0)]^{\mathrm{bos}}_{\mathrm{div}}=\left(\hat e^2\frac{\hat s^2-\hat c^2}{\hat s^2\hat c^2}\hat M_W^2+4e_H^2g_E^2\hat M_W^2\right)\eta_{\mathrm{div}} \tag{D.2}$$

(we have used ${c'}^2\hat M_{Z^0}^2+{s'}^2\hat M_{Z'}^2=\hat M_W^2/\hat c^2$). It differs from the SM only by the last term.

Below we list all bosonic contributions to $\Pi_{Z_1Z_2}(q^2)$ for $Z_1Z_2=Z^0Z^0$, $Z'Z'$, Z^0Z':

$$W^+W^-:-\hat e^2\frac{\hat c^2}{\hat s^2}\left[8\tilde A\left(q^2,\hat M_W,\hat M_W\right)+\left(4q^2+2\hat M_W^2\right)\times b_0\left(q^2,\hat M_W,\hat M_W\right)-\frac{2}{3}\frac{q^2}{16\pi^2}\right]\times\begin{pmatrix}c'^2\\ s'^2\\ -c's'\end{pmatrix}\,,$$

$$G^\pm W^\mp:+2\hat M_W^2 b_0\left(q^2,\hat M_W,\hat M_W\right)\times\begin{pmatrix}\left(-\hat e\frac{\hat s}{\hat c}c'+2e_Hg_Es'\right)^2\\ \left(\hat e\frac{\hat s}{\hat c}s'+2e_Hg_Ec'\right)^2\\ \left(-\hat e\frac{\hat s}{\hat c}c'+2e_Hg_Es'\right)\left(\hat e\frac{\hat s}{\hat c}s'+2e_Hg_Ec'\right)\end{pmatrix}\,,$$

$$G^+G^-:-\tilde A\left(q^2,\hat M_W,\hat M_W\right)\times\begin{pmatrix}\left(\hat e\frac{\hat c^2-\hat s^2}{\hat s\hat c}c'+2e_Hg_Es'\right)^2\\ \left(-\hat e\frac{\hat c^2-\hat s^2}{\hat s\hat c}s'+2e_Hg_Ec'\right)^2\\ \left(\hat e\frac{\hat c^2-\hat s^2}{\hat s\hat c}c'+2e_Hg_Es'\right)\left(-\hat e\frac{\hat c^2-\hat s^2}{\hat s\hat c}s'+2e_Hg_Ec'\right)\end{pmatrix}\,,$$

$$G^0h^0:-\tilde A\left(q^2,\hat M_{Z^0},\hat M_{h^0}\right)\times\begin{pmatrix}\left(\frac{\hat e}{\hat s\hat c}c'-2e_Hg_Es'\right)^2\\ \left(\frac{\hat e}{\hat s\hat c}s'+2e_Hg_Ec'\right)^2\\ \left(-\frac{\hat e}{\hat s\hat c}c'+2e_Hg_Es'\right)\left(\frac{\hat e}{\hat s\hat c}s'+2e_Hg_Ec'\right)\end{pmatrix}\,,$$

$$G'S^0:-4\tilde A\left(q^2,\hat M_{Z'},\hat M_{S^0}\right)\times\begin{pmatrix}e_S^2g_E^2s'^2\\ e_S^2g_E^2c'^2\\ e_S^2g_E^2c's'\end{pmatrix}\,,$$

$$Z^0h^0:+\frac{1}{4}\hat v_H^2 b_0\left(q^2,\hat M_{Z^0},\hat M_{h^0}\right)\times\begin{pmatrix}\left(-\frac{\hat e}{\hat s\hat c}c'+2e_Hg_Es'\right)^4\\ \left(-\frac{\hat e}{\hat s\hat c}c'+2e_Hg_Es'\right)^2\left(\frac{\hat e}{\hat s\hat c}s'+2e_Hg_Ec'\right)^2\\ \left(-\frac{\hat e}{\hat s\hat c}c'+2e_Hg_Es'\right)^3\left(\frac{\hat e}{\hat s\hat c}s'+2e_Hg_Ec'\right)\end{pmatrix}\,,$$

$$Z'h^0:+\frac{1}{4}\hat v_H^2 b_0\left(q^2,\hat M_{Z'},\hat M_{h^0}\right)$$

$$\times \begin{pmatrix} \left(-\frac{\hat{e}}{\hat{s}\hat{c}}c' + 2e_H g_E s'\right)^2 \left(\frac{\hat{e}}{\hat{s}\hat{c}}s' + 2e_H g_E c'\right)^2 \\ \left(\frac{\hat{e}}{\hat{s}\hat{c}}s' + 2e_H g_E c'\right)^4 \\ \left(-\frac{\hat{e}}{\hat{s}\hat{c}}c' + 2e_H g_E s'\right) \left(\frac{\hat{e}}{\hat{s}\hat{c}}s' + 2e_H g_E c'\right)^3 \end{pmatrix},$$

$$Z^0 S^0 : +4\hat{v}_S^2 e_S^4 g_E^4 b_0\left(q^2, \hat{M}_{Z^0}, \hat{M}_{S^0}\right) \times \begin{pmatrix} s'^4 \\ c'^2 s'^2 \\ c' s'^3 \end{pmatrix},$$

$$Z' S^0 : +4\hat{v}_S^2 e_S^4 g_E^4 b_0\left(q^2, \hat{M}_{Z'}, \hat{M}_{S^0}\right) \times \begin{pmatrix} c'^2 s'^2 \\ c'^4 \\ c'^3 s' \end{pmatrix}.$$

To simplify the calculations we have assumed here that the scalar fields H and S do not mix in the potential, so that the Higgs boson h^0 comes only from the doublet H, and S^0 comes only from the singlet S^0.

The fermion contribution to $\Pi_{Z_1 Z_2}(q^2)$ reads

$$\begin{aligned} \Pi^{\rm ferm}_{Z^i Z^j}(q^2) = \sum_f N_c^{(f)} \Big\{ & 2\left(c^{Z^i}_{f\rm L} c^{Z^j}_{f\rm R} + c^{Z^i}_{f\rm R} c^{Z^j}_{f\rm L}\right) m_f^2 \\ & \times b_0\left(q^2, m_f, m_f\right) + \left(c^{Z^i}_{f\rm L} c^{Z^j}_{f\rm L} + c^{Z^i}_{f\rm R} c^{Z^j}_{f\rm R}\right) \\ & \times \left[4\tilde{A}\left(q^2, m_f, m_f\right) + \left(q^2 - 2m_f^2\right) \right. \\ & \left. \times\; b_0\left(q^2, m_f, m_f\right)\right] \Big\}, \end{aligned} \tag{D.3}$$

where N_c is the color factor and the couplings $c^{Z^i}_{f\rm L}$, $c^{Z^i}_{f\rm R}$ can be read off from (20) and (21).

Appendix E: Loop functions

Here we define some loop functions to make the calculations presented in the text complete. We have

$$16\pi^2 a(m) = m^2 \left(\eta_{\rm div} - 1 + \ln\frac{m^2}{\mu^2}\right), \tag{E.1}$$

$$\begin{aligned} 16\pi^2 b_0(q^2, m_1, m_2) = \eta_{\rm div} + & \int_0^1 {\rm d}x \\ & \times \ln\frac{q^2 x(x-1) + x m_1^2 + (1-x)m_2^2}{\mu^2}, \end{aligned} \tag{E.2}$$

$$\begin{aligned} 16\pi^2 b_0(0, m_1, m_2) = \eta_{\rm div} - 1 + & \frac{m_1^2}{m_1^2 - m_2^2}\ln\frac{m_1^2}{\mu^2} \\ + & \frac{m_2^2}{m_2^2 - m_1^2}\ln\frac{m_2^2}{\mu^2}, \end{aligned} \tag{E.3}$$

$$\begin{aligned} \tilde{A}\left(q^2, m_1, m_2\right) = & -\frac{1}{6}a(m_1) - \frac{1}{6}a(m_2) \\ & + \frac{1}{6}\left(m_1^2 + m_2^2 - \frac{q^2}{2}\right) b_0\left(q^2, m_1, m_2\right) \\ & + \frac{m_1^2 - m_2^2}{12q^2}\left[a(m_1) - a(m_2) - \left(m_1^2 - m_2^2\right)\right. \\ & \left. \times\, b_0\left(q^2, m_1, m_2\right)\right] \\ & - \frac{1}{16\pi^2}\frac{1}{6}\left(m_1^2 + m_2^2 - \frac{q^2}{3}\right). \end{aligned} \tag{E.4}$$

The divergent part of $\tilde{A}(q^2, m_1, m_2)$ is

$$16\pi^2\left[\tilde{A}\left(q^2, m_1, m_2\right)\right]_{\rm div} = -\frac{1}{12}q^2\eta_{\rm div}, \tag{E.5}$$

and $\tilde{A}(0, m_1, m_2)$ is finite and reads

$$\begin{aligned} 16\pi^2\tilde{A}(0, m_1, m_2) & = -\frac{1}{8}\left[m_1^2 + m_2^2 - \frac{2m_1^2 m_2^2}{m_1^2 - m_2^2}\log\frac{m_1^2}{m_2^2}\right] \\ & \equiv -\frac{1}{8}g(m_1, m_2). \end{aligned} \tag{E.6}$$

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